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Variational solutions to the total variation flow on metric measure spaces

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ABSTRACT

We discuss a purely variational approach to the total variation flow on metric measure spaces with a doubling measure and a Poincaré inequality. We apply the concept of parabolic De Giorgi classes together with upper gradients, Newtonian spaces and functions of bounded variation to prove a necessary and sufficient condition for a variational solution to be continuous at a given point.

1. Introduction

The total variation flow (TVF) is the partial differential equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 \quad \text{on} \quad \Omega_T = \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^N$ is an open set and $T > 0$. There are several ways to define the concept of weak solution to the TVF. One possibility is to apply the so-called Anzellotti pairing [5]. This approach has been applied in existence and uniqueness results for the total variation flow in [2–4,6]. The variational inequality related to the TVF is

$$\int_0^T \|Du(t)\|(\Omega)\,\mathrm{d}t - \int_0^T \int_\Omega u(t)\frac{\partial \varphi}{\partial t}(t)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \|D(u-\varphi)(t)\|(\Omega)\,\mathrm{d}t$$

for every $\varphi \in C_0^\infty(\Omega_T)$, where the total variation $\|Du(t)\|(\Omega)$ is a Radon measure for almost every $t \in (0,T)$. A variational approach to existence and uniqueness questions has been discussed by Bögelein, Duzaar and

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Marcellini [8], see also Bögelein, Duzaar and Scheven [10], and for the corresponding obstacle problem in [9]. The natural function space for weak solutions to the TVF is bounded variation (BV). Functions of bounded variation on metric measure spaces have been studied in [1,38]. Instead of partial derivatives, this approach is based on the modulus of the gradient and using concepts such as minimal upper gradients and Newtonian spaces, see [7,22–24,30,31,40]. This is also the main advantage of the variational approach. A central motivation for developing such a theory has been the desire to unify the assumptions and methods employed in various specific spaces, such as Riemannian manifolds, Heisenberg groups, graphs, etc.

The regularity theory of nonlinear parabolic problems in the metric space context has been developed and studied in [25,26,28,34–37]. The comparison principle has been discussed in [29] and stability theory has been investigated in [17–19]. Existence for parabolic problems on metric spaces has been discussed in [13]. All of these results consider variational inequalities with p-growth for $p > 1$. For the case $p = 1$, which corresponds to the TVF, Buffa, Collins and Pacchiano [12] showed existence of a parabolic minimizer using the concept of global variational solution. Górny and Mazón [21] studied the existence and uniqueness of weak solutions of the Neumann and Dirichlet problems to the TVF in metric measure spaces. The main goal of the present paper is to extend the results of DiBenedetto, Gianazza and Klaus [15] to a metric measure space with a doubling measure and a Poincaré inequality, see also [20]. The main result gives a necessary and sufficient condition for a variational solution to be continuous at a given point, see Theorem 7.1. Our assumption on the time regularity of a variational solution is initially weaker than in [15] and thus our results may be interesting also in the Euclidean case. As far as we know, this is the first time when regularity questions are discussed for parabolic problems with linear growth on metric measure spaces.

The first step is to derive an energy estimate for variational solutions, in other words, to prove that variational solutions belong to a parabolic De Giorgi class, see Proposition 4.2. The regularity results are based only on this energy estimate and on the assumptions made on the underlying metric measure space, but there is a technical difficulty present when establishing energy estimates for variational solutions. It is not clear that the time regularity of a variational solution is a priori sufficient for placing it as the test function and performing the usual techniques used for obtaining an energy estimate. We resolve this issue by using a mollification technique. The idea of this technique is to prove the required energy estimate for mollified functions and finally to conclude the estimate at the limit. To establish the limiting estimate, we consider the upper gradient of a difference of functions, see Lemma 2.3. In the Euclidean case this poses no difficulties as we can use the linearity of the gradients, in the general metric setting the situation is not that simple, as taking an upper gradient is not a linear operation.

This paper is organized as follows. In Section 2 we recall basic definitions and describe the general setup of our study. Several results related to the function spaces and Sobolev–Poincaré inequalities may be of independent interest. In Section 3 we concentrate on the definition and properties of a variational solution to the TVF. Section 4 explores the relationship between variational solutions to the TVF and the parabolic De Giorgi classes. In Sections 5 and 6, respectively, we prove that functions in a parabolic De Giorgi class are locally bounded and give a time expansion of positivity result. Finally, in Section 7 we present the characterization of continuity, i.e. we prove necessary and sufficient conditions for a variational solution to the TVF to be continuous at a given point. The last three sections are extensions of the corresponding results on Euclidean spaces by DiBenedetto, Gianazza and Klaus in [15] to metric measure spaces.

2. Preliminaries

2.1. NewtonIan spaces

Let (X, d, μ) be a complete metric measure space endowed with a Borel measure μ. The measure μ is said to satisfy the doubling condition if there exists a constant $C_\mu \geq 1$, called the doubling constant of μ, such that

$$0 < \mu(B_{2r}(x)) \le C_\mu \mu(B_r(x)) < \infty, \tag{2.1}$$

for every $x \in X$ and $r > 0$. Here $B_r(x) = \{y \in X : d(x,y) < r\}$ is an open ball centered at $x \in X$ with radius $r > 0$. We assume throughout that the measure μ is nontrivial in the sense that $0 < \mu(B_r(x)) < \infty$ for every $x \in X$ and $r > 0$. A complete metric measure space with a doubling measure is proper, that is, closed and bounded subsets are compact, see [7, Proposition 3.1]. The doubling condition implies that for any $x \in X$, we have

$$\frac{\mu(B_R(x))}{\mu(B_r(x))} \le C\left(\frac{R}{r}\right)^Q, \tag{2.2}$$

for $0 < r < R$ with $Q = \log_2 C_\mu$ and $C = C_\mu^{-2}$, see [7, Lemma 3.3]. The exponent $Q = \log_2 C_\mu$ is sometimes called the homogeneous dimension of (X, d, μ).

A path γ is a continuous mapping from a compact subinterval of $\mathbb{R}$ to X. The p-modulus, with $1 \le p < \infty$, of a path family Γ on X is

$$\mathrm{Mod}_p(\Gamma) = \inf \int_X \rho^p \,\mathrm{d}\mu,$$

where the infimum is taken over all nonnegative Borel functions ρ with $\int_\gamma \rho \mathrm{d}s \ge 1$ for all $\gamma \in \Gamma$, see [7, Section 1.5]. We recall the definition of upper gradient introduced and studied by [23,31] and [40]. General references for this theory are [7,22] and [24].

Definition 2.1. A nonnegative Borel function g on X is an upper gradient of a function $u : X \to [-\infty, \infty]$ if for all paths γ in X, we have

$$|u(x) - u(y)| \le \int_\gamma g \,\mathrm{d}s, \tag{2.3}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g \,\mathrm{d}s = \infty$ otherwise. Here x and y are the endpoints of γ. Moreover, if a nonnegative μ-measurable function g satisfies (2.3) for p-almost every path, that is with the exception of a path family of zero p-modulus, then g is called a p-weak upper gradient of u.

For $1 \le p < \infty$ and an open set $\Omega \subset X$, let

$$\|u\|_{N^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \inf \|g\|_{L^p(\Omega)},$$

where the infimum is taken over all upper gradients g of u. Consider the collection of all functions $u \in L^p(\Omega)$ with an upper gradient $g \in L^p(\Omega)$ and let

$$\widetilde{N}^{1,p}(\Omega) = \{u : \|u\|_{N^{1,p}(\Omega)} < \infty\}.$$

The Newtonian space is defined by

$$N^{1,p}(\Omega) = \{u : \|u\|_{N^{1,p}(\Omega)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(\Omega)} = 0$.

The corresponding local Newtonian space is defined by $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(\Omega')$ for all $\Omega' \Subset \Omega$, see [7, Proposition 2.29]. Here $\Omega' \Subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of Ω. If u has an upper gradient $g \in L^p(\Omega)$, there exists a unique minimal p-weak upper gradient $g_u \in L^p(\Omega)$ with $g_u \le g$ μ-almost everywhere for all p-weak upper gradients $g \in L^p(\Omega)$ of u, see [7, Theorem 2.5]. Moreover, the minimal p-weak upper gradient is unique up to sets of measure zero. For $u \in N^{1,p}(\Omega)$ we have

$$\|u\|_{N^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|g_u\|_{L^p(\Omega)},$$

where g_u is the minimal p-weak upper gradient of u. The main advantage is that p-weak upper gradients behave better under L^p-convergence than upper gradients, see [7, Proposition 2.2]. However, the difference is relatively small, since every p-weak upper gradient can be approximated be a sequence of upper gradients in L^p, see [7, Lemma 1.46]. This implies that the $N^{1,p}$-norm above remains the same if the infimum is taken over upper gradients instead of p-weak upper gradients.

We collect some calculus rules for upper gradients on metric measure spaces. Let $u, v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and let g_u, $g_v \in L^p_{\mathrm{loc}}(\Omega)$ be the p-weak upper gradients of u and v, respectively. Then $g_u + g_v$ and $|u|g_v + |v|g_u$ are p-weak upper gradients for $u + v$ and uv, respectively, see [7, Theorem 2.15]. Let η be Lipschitz continuous on Ω with $0 \le \eta \le 1$ and consider $w = u + \eta(u - v) = (1 - \eta)u + \eta v$. Then $(1 - \eta)g_u + \eta g_v + |v - u|g_\eta$ is a p-weak upper gradient of w, see [7, Theorem 2.18]. Moreover, $g_u = g_v$, μ-almost everywhere on the set $\{x \in X : u(x) = v(x)\}$. In particular, if $c \in \mathbb{R}$ is a constant, then $g_u = 0$ μ-almost everywhere on the set $\{x \in X : u(x) = c\}$, see [7, Corollary 2.21].

A metric measure space (X, d, μ) supports a weak Poincaré inequality, if there exist a constant C_P and a dilation factor $\tau \ge 1$ such that for every ball $B_\rho(x_0)$ in X, for every $u \in L^1_{\mathrm{loc}}(X)$ and every upper gradient g of u, we have

$$\fint_{B_\rho(x_0)} |u - u_{B_\rho(x_0)}| \,\mathrm{d}\mu \le C_P \rho \fint_{B_{\tau\rho}(x_0)} g \,\mathrm{d}\mu, \tag{2.4}$$

where the integral average is denoted by

$$u_{B_\rho(x_0)} = \fint_{B_\rho(x_0)} u \,\mathrm{d}\mu = \frac{1}{\mu(B_\rho(x_0))} \int_{B_\rho(x_0)} u \,\mathrm{d}\mu.$$

A space supporting a Poincaré inequality is connected, see [7, Proposition 4.2]. Throughout the work, we assume that the measure μ is doubling and that the metric measure space (X, d, μ) supports a weak Poincaré inequality. The weak Poincaré inequality and the doubling condition imply the Sobolev–Poincaré inequality

$$\left(\fint_{B_\rho(x_0)} |u - u_{B_\rho(x_0)}|^{\frac{Q}{Q-1}} \,\mathrm{d}\mu \right)^{\frac{Q-1}{Q}} \le C\rho \fint_{B_{2\tau\rho}(x_0)} g \,\mathrm{d}\mu, \tag{2.5}$$

for every $u \in L^1_{\mathrm{loc}}(X)$ and every 1-weak upper gradient g of u and for every ball $B_\rho(x_0)$ in X with $C = C(C_\mu, C_P)$ and Q as in (2.2), see [7, Theorem 4.21].

Next we discuss parabolic Newtonian spaces.

Definition 2.2. Let $\Omega \subset X$ be an open set, $0 < T < \infty$ and $1 \le p < \infty$. The parabolic Newtonian space $L^p(0, T; N^{1,p}(\Omega))$ consists of strongly measurable functions $u : (0, T) \to N^{1,p}(\Omega)$ with the norm

$$\|u\|_{L^p(0,T;N^{1,p}(\Omega))} = \left(\int_0^T \|u(t)\|^p_{N^{1,p}(\Omega)} \,\mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

The integration over $(0, T)$ is taken with respect to the one-dimensional Lebesgue measure $\mathcal{L}^1$. We say that $u \in L^p_{\mathrm{loc}}(0, T; N^{1,p}_{\mathrm{loc}}(\Omega))$ if for every $\Omega' \times (t_1, t_2) \Subset \Omega_T$ we have $u \in L^p(t_1, t_2; N^{1,p}(\Omega'))$. Moreover, we denote $u \in L^p_c(0, T; N^{1,p}(\Omega))$ if for some $0 < t_1 < t_2 < T$ we have $u(t) = 0$ outside $[t_1, t_2]$.

The strong measurability of $u : (0, T) \to N^{1,p}(\Omega)$ and the assumption $u \in L^p(0, T; N^{1,p}(\Omega))$, imply that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ of simple functions $u_k : (0, T) \to N^{1,p}(\Omega)$,

$$u_k(t) = \sum_{i=1}^{n_k} \mathbb{1}_{E_i^{(k)}}(t) \cdot u_i^{(k)}, \tag{2.6}$$

where $\{E_i^{(k)}\}_{i=1}^{n_k}$ is a $\mathcal{L}^1$-measurable pairwise disjoint partition of $(0,T)$ and $v_i^{(k)} \in N^{1,p}(\Omega)$, $i = 1,\dots,n_k$, such that $u_k \to u$ in $L^p(0,T;N^{1,p}(\Omega))$ as $k \to \infty$. In particular, we have $u_k(t) \to u(t)$ in $N^{1,p}(\Omega)$ for $\mathcal{L}^1$-almost every $t \in (0,T)$. In other words, up to relabeling, we have

$$u(t) = \sum_{k=1}^{\infty} \mathbb{1}_{E_k}(t) \cdot u_k, \tag{2.7}$$

with the sets E_k and simple functions u_k as in (2.6).

Next we consider upper gradients. Since $u_k(t) \to u(t)$ in $N^{1,p}(\Omega)$ for $\mathcal{L}^1$-almost every $t \in (0,T)$ as $k \to \infty$, we have $u(t) \in N^{1,p}(\Omega)$ for $\mathcal{L}^1$-almost every $t \in (0,T)$. Consider the minimal p-weak upper gradient $g_{u(t)} \in L^p(\Omega)$ of $u(t)$ for $\mathcal{L}^1$-almost every $t \in (0,T)$. The parabolic p-weak upper gradient of $u \in L^p(0,T;N^{1,p}(\Omega))$ is defined to be $g_u = g_{u(t)}$ for $\mathcal{L}^1$-almost every $t \in (0,T)$.

We note that the function g_u is strongly measurable. For $\mathcal{L}^1$-almost every $t \in (0,T)$, the function $u(t)$ is the limit of strongly measurable functions $u_k(t)$ defined in (2.6). By (2.7) and the locality of minimal p-weak upper gradients, we have

$$g_{u(t)} = g_{\sum_{k=1}^{\infty} \mathbb{1}_{E_k}(t)\cdot u_k} = \sum_{k=1}^{\infty} \mathbb{1}_{E_k}(t) \cdot g_{u_k}, \tag{2.8}$$

for $\mathcal{L}^1$-almost every $t \in (0,T)$. Strong measurability follows, since $u_k \in N^{1,p}(\Omega)$ and $g_{u_k} \in L^p(\Omega)$ for every $k \in \mathbb{N}$. In other words, $g_{u(t)}$ can be approximated in $L^p(\Omega)$ by the functions $g_{u_k}(t) \in L^p(\Omega)$,

$$g_{u_k(t)} = g_{\sum_{i=1}^{n_k} \mathbb{1}_{E_i^{(k)}}(t)\cdot u_i^{(k)}} = \sum_{i=1}^{n_k} \mathbb{1}_{E_i^{(k)}}(t) \cdot g_{u_i^{(k)}},$$

which we obtain from (2.6) by arguing as in (2.8). Since $u_k \to u$ in $L^p(0,T;N^{1,p}(\Omega))$, we have $u_k \to u$ in $L^p(0,T;L^p(\Omega))$ and $g_{u_k} \to g_u$ in $L^p(0,T;L^p(\Omega))$ as $k \to \infty$.

The product measure in the space $X \times (0,T)$, $T > 0$, is denoted by $\mu \otimes \mathcal{L}^1$. For $T > 0$, we denote the space–time cylinder over an open subset $\Omega \subset X$ as $\Omega_T = \Omega \times (0,T)$. For $u \in L^p(0,T;L^p(\Omega))$, there exists a $(\mu \otimes \mathcal{L}^1)$-measurable representative $u : \Omega_T \to [-\infty,\infty]$ such that $u(t) = u(\cdot,t)$ for $\mathcal{L}^1$-almost every $t \in (0,T)$ and

$$\int_0^T \int_\Omega |u(x,t)|^p \,\mathrm{d}\mu \,\mathrm{d}t = \int_0^T \|u(t)\|^p_{L^p(\Omega)} \,\mathrm{d}t.$$

See [32, Theorem 23.21] and [39, Section 2.1.1]). Similarly, for $g_u \in L^p(0,T;L^p(\Omega))$, there exists a $(\mu \otimes \mathcal{L}^1)$-measurable representative $g_u : \Omega_T \to [-\infty,\infty]$ such that $g_u(t) = g_u(\cdot,t)$ for $\mathcal{L}^1$-almost every $t \in (0,T)$.

With these observations we may consider the parabolic Newtonian space $L^p(0,T;N^{1,p}(\Omega))$ to be the space of functions $u \in L^p(\Omega_T)$, with $u = u(x,t)$, such that for $\mathcal{L}^1$-almost every $t \in (0,T)$ the function $u(\cdot,t)$ belongs to $N^{1,p}(\Omega)$ and there exists $g_u \in L^p(\Omega_T)$ such that for $\mathcal{L}^1$-almost every $t \in (0,T)$ the function $g_u(\cdot,t)$ is a minimal p-weak upper gradient of $u(\cdot,t)$ with

$$\iint_{\Omega_T} (|u(x,t)|^p + |g_u(x,t)|^p) \,\mathrm{d}\mu \,\mathrm{d}t < \infty.$$

Let $u \in L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$, $1 \le p < \infty$, and consider the time mollification

$$u_\varepsilon(t) = \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(s) u(t-s) \,\mathrm{d}s,$$

where $\eta_\varepsilon(s) = \frac{1}{s}\eta(\frac{s}{\varepsilon})$, $\varepsilon > 0$, is a standard mollifier. The following approximation result was proved in more generality in [11]. We include a slightly modified version together with its full proof for reader's convenience. We say that $u_\varepsilon \to u$ in $L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$, if $\|u_\varepsilon - u\|_{L^p(t_1,t_2;N^{1,p}(\Omega'))} \to 0$ as $\varepsilon \to 0$ for every $\Omega' \times (t_1,t_2) \Subset \Omega_T$, where $\Omega' \Subset \Omega$ and $0 < t_1 < t_2 < T$.

Lemma 2.3. *Let $\Omega \subset X$ be an open set and assume that $u \in L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$, $1 \le p < \infty$. Then $u_\varepsilon \to u$ in $L^p_{\mathrm{loc}}(0,T;N^{1,p}_{\mathrm{loc}}(\Omega))$ as $\varepsilon \to 0$. In particular, we have $g_{u_\varepsilon - u} \to 0$ in $L^p_{\mathrm{loc}}(\Omega_T)$ as $\varepsilon \to 0$. Moreover, as $s \to 0$, we have $g_{u(\cdot,t-s)-u(\cdot,t)} \to 0$ in $L^p_{\mathrm{loc}}(\Omega_T)$ uniformly in t.*

Proof. Since

$$u(t) = \sum_{k=1}^{\infty} \mathbb{1}_{E_k}(t) \cdot u_k,$$

for $\mathcal{L}^1$-almost every $t \in (0,T)$, by the definition of the time mollification we have

$$\begin{aligned} u_\varepsilon(t) &= \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(s) u(t-s)\,\mathrm{d}s = \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(s) \sum_{k=1}^{\infty} \mathbb{1}_{E_k}(t-s) \cdot u_k \,\mathrm{d}s \\ &= \sum_{k=1}^{\infty} \left(\int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(s) \mathbb{1}_{E_k}(t-s)\,\mathrm{d}s \right) \cdot u_k = \sum_{k=1}^{\infty} (\mathbb{1}_{E_k})_\varepsilon(t) \cdot u_k, \end{aligned}$$

which implies

$$u(t) - u_\varepsilon(t) = \sum_{k=1}^{\infty} \big(\mathbb{1}_{E_k}(t) - (\mathbb{1}_{E_k})_\varepsilon(t) \big) \cdot u_k,$$

for $\mathcal{L}^1$-almost every $t \in (0,T)$. By a standard mollifier argument we conclude that $u_\varepsilon \to u$ in $L^p_{\mathrm{loc}}(\Omega_T)$ as $\varepsilon \to 0$.

By properties of minimal p-weak upper gradients and standard mollifications, we obtain

$$\begin{aligned} g_{u-u_\varepsilon} &= g_{\sum_{k=1}^{\infty} (\mathbb{1}_{E_k} - (\mathbb{1}_{E_k})_\varepsilon) \cdot u_k} \le \sum_{k=1}^{\infty} g_{(\mathbb{1}_{E_k} - (\mathbb{1}_{E_k})_\varepsilon) \cdot u_k} \\ &= \sum_{k=1}^{\infty} \big| \mathbb{1}_{E_k} - (\mathbb{1}_{E_k})_\varepsilon \big| \cdot g_{u_k} \xrightarrow{\varepsilon \to 0} 0, \end{aligned}$$

$\mathcal{L}^1$-almost everywhere on $(0,T)$. It follows that $g_{u-u_\varepsilon} \to 0$ $\mu \otimes \mathcal{L}^1$-almost everywhere in Ω_T as $\varepsilon \to 0$. Again, a standard mollifier argument implies $g_{u-u_\varepsilon} \to 0$ in $L^p_{\mathrm{loc}}(\Omega_T)$ as $\varepsilon \to 0$.

It remains to prove that $g_{u(t-s)-u(t)} \to 0$ in $L^p_{\mathrm{loc}}(\Omega_T)$ as $s \to 0$, uniformly in t. As above, we have

$$g_{u(t-s)-u(t)} = \sum_{k=1}^{\infty} \big| \mathbb{1}_{E_k}(t-s) - \mathbb{1}_{E_k}(t) \big| \cdot g_{u_k},$$

for every $s > 0$.

Let $\Omega' \times (t_1,t_2) \Subset \Omega_T$. By Fubini's theorem and Minkowski's inequality, we have

$$\begin{aligned} \left(\int_{\Omega' \times (t_1,t_2)} g^p_{u(t-s)-u(t)} \,\mathrm{d}\mu\,\mathrm{d}t \right)^{\frac{1}{p}} &= \left(\int_{\Omega' \times (t_1,t_2)} \left(\sum_{k=1}^{\infty} \big| \mathbb{1}_{E_k}(t-s) - \mathbb{1}_{E_k}(t) \big| \cdot g_{u_k} \right)^p \mathrm{d}\mu\,\mathrm{d}t \right)^{\frac{1}{p}} \\ &\le \left(\int_{t_1}^{t_2} \int_{\Omega'} \left(\sum_{k=1}^{\infty} \big| \mathbb{1}_{E_k}(t-s) - \mathbb{1}_{E_k}(t) \big| \right)^p \left(\sum_{k=1}^{\infty} g_{u_k} \right)^p \mathrm{d}\mu\,\mathrm{d}t \right)^{\frac{1}{p}} \\ &= \left(\int_{t_1}^{t_2} \left(\sum_{k=1}^{\infty} \big| \mathbb{1}_{E_k}(t-s) - \mathbb{1}_{E_k}(t) \big| \right)^p \mathrm{d}t \right)^{\frac{1}{p}} \left(\int_{\Omega'} \left(\sum_{k=1}^{\infty} g_{u_k} \right)^p \mathrm{d}\mu \right)^{\frac{1}{p}} \\ &= \left\| \sum_{k=1}^{\infty} \big(\mathbb{1}_{E_k}(t-s) - \mathbb{1}_{E_k}(t) \big) \right\|_{L^p((t_1,t_2))} \left\| \sum_{k=1}^{\infty} g_{u_k} \right\|_{L^p(\Omega')} \end{aligned}$$

$$\le \sum_{k=1}^{\infty} \|\mathbb{1}_{E_k}(t-s) - \mathbb{1}_{E_k}(t)\|_{L^p((t_1,t_2))} \cdot \|g_{u_k}\|_{L^p(\Omega')}$$
$$= \sum_{k=1}^{\infty} \left(\int_{t_1}^{t_2} \left|\mathbb{1}_{E_k}(t-s) - \mathbb{1}_{E_k}(t)\right|^p \mathrm{d}t \right)^{\frac{1}{p}} \left(\int_{\Omega'} g_{u_k}^p \,\mathrm{d}\mu \right)^{\frac{1}{p}}.$$

Since $\mathbb{1}_{E_k} \in L^p(0,T)$, $k \in \mathbb{N}$, the expression above vanishes as $s \to 0$ by the continuity of translations on L^p functions. $\square$

2.2. BV Functions

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [38]. The total variation of $u \in L^1_{\mathrm{loc}}(X)$ is defined as

$$\|Du\|(X) = \inf\left\{ \liminf_{i\to\infty} \int_X g_{u_i} \,\mathrm{d}\mu \right\},$$

where the infimum is taken over all sequences $(u_i)_{i\in\mathbb{N}}$ with $u_i \in \mathrm{Lip}_{\mathrm{loc}}(X)$ for every $i \in \mathbb{N}$ and $u_i \to u$ in $L^1_{\mathrm{loc}}(X)$ as $i \to \infty$. Here g_{u_i} is a 1-weak upper gradient of u_i and $\mathrm{Lip}_{\mathrm{loc}}(X)$ denotes the class of functions that are Lipschitz continuous on compact subsets of X. We say that a function $u \in L^1(X)$ is of bounded variation, and denote $u \in BV(X)$, if $\|Du\|(X) < \infty$. By replacing X with an open set $\Omega \subset X$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. A function $u \in BV_{\mathrm{loc}}(\Omega)$ if $u \in BV(\Omega')$ for all open sets $\Omega' \Subset \Omega$. For an arbitrary set $A \subset X$, we set

$$\|Du\|(A) = \inf\{\|Du\|(U) : A \subset U,\ U \subset X \text{ is open}\}.$$

If $u \in BV(\Omega)$, then $\|Du\|(A)$ is a finite Radon measure on Ω by [38, Theorem 3.4]. For the following result, see [33, Theorem 4.3].

Theorem 2.4. *Let $\Omega \subset X$ be an open set and $u \in L^1_{\mathrm{loc}}(\Omega)$. If $\|Du\|(\Omega) < \infty$, then*

$$\|Du\|(\Omega) = \inf\left\{ \liminf_{i\to\infty} \int_\Omega g_{u_i} \,\mathrm{d}\mu : u_i \in N^{1,1}_{\mathrm{loc}}(\Omega),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \text{ as } i \to \infty \right\},$$

where g_{u_i} is the minimal 1-weak upper gradient of u_i in Ω.

If the space supports the Poincaré inequality in (2.4), by an approximation argument, for every $u \in BV_{\mathrm{loc}}(X)$ and every ball $B_\rho(x_0)$ in X, we have

$$\begin{aligned} \fint_{B_\rho(x_0)} |u - u_{B_\rho(x_0)}| \,\mathrm{d}\mu &\le \left(\fint_{B_\rho(x_0)} |u - u_{B_\rho(x_0)}|^{\frac{Q}{Q-1}} \,\mathrm{d}\mu \right)^{\frac{Q-1}{Q}} \\ &\le C\rho \frac{\|Du\|(B_{2\tau\rho}(x_0))}{\mu(B_{2\tau\rho}(x_0))}, \end{aligned} \tag{2.9}$$

where the constant C and the dilation factor τ are the same as in the Sobolev–Poincaré inequality in (2.5). Next we state a Sobolev type inequality for BV functions which vanish on a large set, see [27] and [7, Theorem 5.51] for the corresponding result for Newtonian spaces.

Theorem 2.5. *Assume that μ is doubling and that (X, d, μ) supports a Poincaré inequality. Then there exists a constant $C = C(C_\mu, C_P)$ such that if $B_\rho(x_0)$ is a ball in X with $0 < \rho < \frac{1}{4}\mathrm{diam}X$ and $u \in BV(X)$*

with $u = 0$ in $X \setminus B_\rho(x_0)$, then

$$\left(\fint_{B_\rho(x_0)} |u|^{\frac{Q}{Q-1}} \,\mathrm{d}\mu \right)^{\frac{Q-1}{Q}} \le C\rho \frac{\|Du\|(B_\rho(x_0))}{\mu(B_\rho(x_0))}, \tag{2.10}$$

where Q is as in (2.2).

Proof. By Minkowski's inequality and (2.9) we have

$$\begin{aligned} \left(\fint_{B_{2\rho}(x_0)} |u|^{\frac{Q}{Q-1}} \,\mathrm{d}\mu \right)^{\frac{Q-1}{Q}} &\le \left(\fint_{B_{2\rho}(x_0)} |u - u_{B_{2\rho}(x_0)}|^{\frac{Q}{Q-1}} \,\mathrm{d}\mu \right)^{\frac{Q-1}{Q}} + |u_{B_{2\rho}(x_0)}| \\ &\le C\rho \frac{\|Du\|(B_{4\tau\rho}(x_0))}{\mu(B_{4\tau\rho}(x_0))} + |u_{B_{2\rho}(x_0)}|. \end{aligned}$$

Hölder's inequality and the fact that $u = 0$ in $B_{2\rho}(x_0) \setminus B_\rho(x_0)$ imply that

$$\begin{aligned} |u_{B_{2\rho}(x_0)}| &\le \fint_{B_{2\rho}(x_0)} |u| \,\mathrm{d}\mu = \fint_{B_{2\rho}(x_0)} |u| \chi_{B_\rho(x_0)} \,\mathrm{d}\mu \\ &\le \left(\frac{\mu(B_\rho(x_0))}{\mu(B_{2\rho}(x_0))} \right)^{\frac{1}{Q}} \left(\fint_{B_{2\rho}(x_0)} |u|^{\frac{Q}{Q-1}} \,\mathrm{d}\mu \right)^{\frac{Q-1}{Q}}. \end{aligned}$$

By [7, Lemma 3.7] we have $\frac{\mu(B_\rho(x_0))}{\mu(B_{2\rho}(x_0))} \le \gamma < 1$, where $\gamma = C(C_\mu)$, and we obtain

$$(1 - \gamma^{\frac{1}{Q}}) \left(\fint_{B_{2\rho}(x_0)} |u|^{\frac{Q}{Q-1}} \,\mathrm{d}\mu \right)^{\frac{Q-1}{Q}} \le C\rho \frac{\|Du\|(B_{4\tau\rho}(x_0))}{\mu(B_{4\tau\rho}(x_0))} \le C\rho \frac{\|Du\|(B_\rho(x_0))}{\mu(B_\rho(x_0))},$$

where we used the fact that $u = 0$ in $B_{4\rho}(x_0) \setminus B_\rho(x_0)$ and thus $\|Du\|(B_{4\rho}(x_0) \setminus B_\rho(x_0)) = 0$. The claim follows since $0 < \gamma < 1$. $\square$

The following isoperimetric inequality in [16, Lemma 2.2] has been originally obtained by De Giorgi. See also [14, Lemma 5.2] for the case $p > 1$. We give a proof that is based on the Sobolev–Poincaré type inequality for BV in (2.9).

Lemma 2.6. *Assume that μ is doubling and that (X, d, μ) supports a Poincaré inequality. Then there exists a constant $C = C(C_\mu, C_P)$ such that if $B_\rho(x_0)$ is a ball in X and $u \in BV_{\mathrm{loc}}(X)$, then for $k < l$ real numbers we get*

$$\frac{(l-k)\mu(B_\rho(x_0) \cap \{u > l\})}{\mu(B_\rho(x_0))} \le \frac{C\rho}{\mu(B_\rho(x_0) \cap \{u \le k\})} \|Du\|(\{k < u < l\}).$$

Proof. Let

$$v = \begin{cases} \min\{u, l\} - k, & \text{if } u > k, \\ 0, & \text{if } u \le k. \end{cases}$$

Notice that

$$\begin{aligned} (l-k)\mu(B_\rho(x_0) \cap \{u > l\})^{\frac{Q}{Q-1}} &= \left(\int_{B_\rho(x_0) \cap \{u > l\}} |v|^{\frac{Q}{Q-1}} \,\mathrm{d}\mu \right)^{\frac{Q-1}{Q}} \\ &\le \left(\int_{B_\rho(x_0)} |v|^{\frac{Q}{Q-1}} \,\mathrm{d}\mu \right)^{\frac{Q-1}{Q}}. \end{aligned}$$

By the Sobolev–Poincaré inequality for BV in (2.9), we have

$$\left(\fint_{B_\rho(x_0)} |v|^{\frac{Q}{Q-1}}\,\mathrm{d}\mu\right)^{\frac{Q-1}{Q}} \le \left(\fint_{B_\rho(x_0)} |v-v_{B_\rho(x_0)}|^{\frac{Q}{Q-1}}\,\mathrm{d}\mu\right)^{\frac{Q-1}{Q}} + \left|v_{B_\rho(x_0)}\right| \le C\rho\frac{\|Dv\|(B_{2\tau\rho}(x_0))}{\mu(B_{2\tau\rho}(x_0))} + \fint_{B_\rho(x_0)} |v|\,\mathrm{d}\mu,$$

where

$$\begin{aligned}\fint_{B_\rho(x_0)} |v|\,\mathrm{d}\mu &= \frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)\cap\{u>k\}} |v|\,\mathrm{d}\mu \\ &\le \frac{1}{\mu(B_\rho(x_0))}\left(\int_{B_\rho(x_0)\cap\{u>k\}} |v|^{\frac{Q}{Q-1}}\,\mathrm{d}\mu\right)^{\frac{Q-1}{Q}} \mu(B_\rho(x_0)\cap\{u>k\})^{\frac{1}{Q}} \\ &= \left(\frac{\mu(B_\rho(x_0)\cap\{u>k\})}{\mu(B_\rho(x_0))}\right)^{\frac{1}{Q}}\left(\fint_{B_\rho(x_0)} |v|^{\frac{Q}{Q-1}}\,\mathrm{d}\mu\right)^{\frac{Q-1}{Q}}.\end{aligned}$$

This implies

$$\left(1-\left(\frac{\mu(B_\rho(x_0)\cap\{u>k\})}{\mu(B_\rho(x_0))}\right)^{\frac{1}{Q}}\right)\left(\fint_{B_\rho(x_0)} |v|^{\frac{Q}{Q-1}}\,\mathrm{d}\mu\right)^{\frac{Q-1}{Q}} \le C\rho\frac{\|Dv\|(B_{2\tau\rho}(x_0))}{\mu(B_{2\tau\rho}(x_0))}. \tag{2.11}$$

On the other hand

$$\begin{aligned}\left(\fint_{B_\rho(x_0)} |v|^{\frac{Q}{Q-1}}\,\mathrm{d}\mu\right)^{\frac{Q-1}{Q}} &\ge (l-k)\left(\frac{\mu(B_\rho(x_0)\cap\{u>l\})}{\mu(B_\rho(x_0))}\right)^{1-\frac{1}{Q}} \\ &\ge \frac{(l-k)\mu(B_\rho(x_0)\cap\{u>l\})}{\mu(B_\rho(x_0))}\left(\frac{\mu(B_\rho(x_0)\cap\{u>k\})}{\mu(B_\rho(x_0))}\right)^{-\frac{1}{Q}}.\end{aligned}$$

By (2.11), we have

$$\left(\left(\frac{\mu(B_\rho(x_0)\cap\{u>k\})}{\mu(B_\rho(x_0))}\right)^{-\frac{1}{Q}}-1\right)\frac{(l-k)\mu(B_\rho(x_0)\cap\{u>l\})}{\mu(B_\rho(x_0))} \le C\rho\frac{\|Dv\|(B_{2\tau\rho}(x_0))}{\mu(B_{2\tau\rho}(x_0))},$$

where, by Bernoulli's inequality, we have

$$\begin{aligned}\left(\frac{\mu(B_\rho(x_0)\cap\{u>k\})}{\mu(B_\rho(x_0))}\right)^{-\frac{1}{Q}} - 1 &= \left(\frac{\mu(B_\rho(x_0))}{\mu(B_\rho(x_0)\cap\{u>k\})}\right)^{\frac{1}{Q}} - 1 \\ &\ge \frac{1}{Q}\left(\frac{\mu(B_\rho(x_0))}{\mu(B_\rho(x_0)\cap\{u>k\})} - 1\right) \\ &= \frac{1}{Q}\frac{\mu(B_\rho(x_0)\cap\{u\le k\})}{\mu(B_\rho(x_0)\cap\{u>k\})} \\ &\ge \frac{1}{Q}\frac{\mu(B_\rho(x_0)\cap\{u\le k\})}{\mu(B_\rho(x_0))}.\end{aligned}$$

This implies

$$\frac{(l-k)\mu(B_\rho(x_0)\cap\{u>l\})}{\mu(B_\rho(x_0))} \le \frac{CQ\rho}{\mu(B_\rho(x_0)\cap\{u\le k\})}\|Dv\|(B_{2\tau\rho}(x_0)),$$

where

$$\begin{aligned}\|Dv\|(B_{2\tau\rho}(x_0)) &= \|Dv\|(B_{2\tau\rho}(x_0)\cap\{k<u<l\}) + \|Dv\|(B_{2\tau\rho}(x_0)\setminus\{k<u<l\})\\ &= \|Dv\|(B_{2\tau\rho}(x_0)\cap\{k<u<l\})\\ &= \|D(u-k)\|(B_{2\tau\rho}(x_0)\cap\{k<u<l\})\\ &= \|Du\|(B_{2\tau\rho}(x_0)\cap\{k<u<l\})\\ &\le \|Du\|(\{k<u<l\}).\end{aligned}$$

This proves the claim. □

We also apply parabolic BV functions.

Definition 2.7. Let $\Omega \subset X$ be an open set and $0 < T < \infty$. We consider a parabolic BV space $L^1(0,T;BV(\Omega))$, which consists of functions $u:(0,T)\to BV(\Omega)$ such that

$$\int_0^T \Big(\|u(t)\|_{L^1(\Omega)} + \|Du(t)\|(\Omega)\Big)\,\mathrm{d}t < \infty.$$

Here

$$\|Du(t)\|(\Omega) = \inf\left\{\liminf_{i\to\infty}\int_\Omega g_{u_i}(t)\,\mathrm{d}\mu\right\},$$

where the infimum is taken over all sequences $(u_i)_{i\in\mathbb{N}}$, with $u_i \in L^1_{\mathrm{loc}}(0,T;N^{1,1}_{\mathrm{loc}}(\Omega))$ for every $i\in\mathbb{N}$ and $u_i \to u$ in $L^1_{\mathrm{loc}}(0,T;N^{1,1}_{\mathrm{loc}}(\Omega))$ as $i\to\infty$. We say that $u \in L^1_{\mathrm{loc}}(0,T;BV_{\mathrm{loc}}(\Omega))$, if for every $\Omega'\times(t_1,t_2)\Subset\Omega_T$, we have $u\in L^1(t_1,t_2;BV(\Omega'))$.

Note that we do not assume strong measurability in the sense of Bochner, which is too restrictive for the parabolic BV theory. The space $L^1(0,T;BV(\Omega))$ satisfies a weaker measurability condition, see [12], which implies that $t\mapsto\|Du(t)\|$ is a Lebesgue measurable function on $(0,T)$. For $u\in L^1(0,T;BV(\Omega))$ there exists a $(\mu\otimes\mathcal{L}^1)$-measurable function $u:\Omega_T\to[-\infty,\infty]$ such that $u(\cdot,t)\in BV(\Omega)$ for $\mathcal{L}^1$-almost every $t\in(0,T)$.

Next we consider a Sobolev inequality for the parabolic BV.

Proposition 2.8. *There exists a constant $C = C(C_\mu, C_P)$, such that if $B_\rho(x_0)$ is a ball in X with $0<\rho<\frac{1}{4}\operatorname{diam}X$ and $u\in L^1_{\mathrm{loc}}(0,T;BV_{\mathrm{loc}}(X))$ with $u=0$ in $(X\setminus B_\rho(x_0))\times(0,T)$, then*

$$\int_{t_1}^{t_2}\int_{B_\rho(x_0)}|u(t)|^\kappa\,\mathrm{d}\mu\,\mathrm{d}t \le C\rho\int_{t_1}^{t_2}\|Du(t)\|(B_\rho(x_0))\,\mathrm{d}t\left(\operatorname*{ess\,sup}_{t_1<t<t_2}\fint_{B_\rho(x_0)}|u(t)|^2\,\mathrm{d}\mu\right)^{\frac{1}{Q}}.$$

where $0<t_1<t_2<T$, $\kappa=\frac{Q+2}{Q}$ and Q is as in (2.2).

Proof. Hölder's inequality and Sobolev's inequality (2.10) imply

$$\begin{aligned}\int_{B_\rho(x_0)}|u(t)|^\kappa\,\mathrm{d}\mu &= \int_{B_\rho(x_0)}|u(t)|^{1+(\kappa-1)}\,\mathrm{d}\mu\\ &\le \left(\int_{B_\rho(x_0)}|u(t)|^{\frac{Q}{Q-1}}\,\mathrm{d}\mu\right)^{\frac{Q-1}{Q}}\left(\int_{B_\rho(x_0)}|u(t)|^{(\kappa-1)Q}\,\mathrm{d}\mu\right)^{\frac{1}{Q}}\\ &= \left(\int_{B_\rho(x_0)}|u(t)|^{\frac{Q}{Q-1}}\,\mathrm{d}\mu\right)^{\frac{Q-1}{Q}}\left(\int_{B_\rho(x_0)}|u(t)|^2\,\mathrm{d}\mu\right)^{\frac{1}{Q}}\\ &\le C\rho\|Du(t)\|(B_\rho(x_0))\left(\fint_{B_\rho(x_0)}|u(t)|^2\,\mathrm{d}\mu\right)^{\frac{1}{Q}},\end{aligned}$$

for $\mathcal{L}^1$-almost every $t\in(0,T)$. The assertion follows by integrating over (t_1,t_2). □

3. Total variation flow

We discuss a definition of a variational solution to the total variation flow.

Definition 3.1. Let $\Omega \subset X$ be an open set and $0 < T < \infty$. A function $u \in L^1_{\mathrm{loc}}(0,T;BV_{\mathrm{loc}}(\Omega))$ is a variational solution to the total variation flow in Ω_T, if

$$\int_0^T \left(\int_\Omega -u(t)\frac{\partial \varphi}{\partial t}(t)\,\mathrm{d}\mu + \|Du(t)\|(\Omega) \right) \mathrm{d}t \le \int_0^T \|D(u+\varphi)(t)\|(\Omega)\,\mathrm{d}t, \tag{3.1}$$

for every $\varphi \in \mathrm{Lip}(\Omega_T)$ with $\operatorname{supp}\varphi \Subset \Omega_T$.

Boundary terms appear for test functions that do not necessarily vanish on the initial and the last moment of time.

Proposition 3.2. *Let $u \in L^1_{\mathrm{loc}}(0,T;BV_{\mathrm{loc}}(\Omega))$ be a variational solution to the total variation flow in Ω_T and let $\Omega' \times (t_1,t_2) \Subset \Omega_T$, where $0 < t_1 < t_2 < T$ are such that the boundary terms below are defined. Then*

$$\int_{t_1}^{t_2} \left(\int_{\Omega'} -u(t)\frac{\partial \varphi}{\partial t}(t)\,\mathrm{d}\mu + \|Du(t)\|(\Omega') \right) \mathrm{d}t \le \int_{t_1}^{t_2} \|D(u+\varphi)(t)\|(\Omega')\,\mathrm{d}t - \left[\int_{\Omega'} u(t)\varphi(t)\,\mathrm{d}\mu \right]_{t=t_1}^{t_2},$$

for every $\varphi \in \mathrm{Lip}(\Omega_T)$ with $\operatorname{supp}\varphi \Subset \Omega' \times (0,T)$.

Proof. Let $\varphi \in \mathrm{Lip}(\Omega_T)$ with supp $\varphi \Subset \Omega' \times (0,T)$. Let ζ_h, $h > 0$, be a cutoff function depending only on time, defined as

$$\zeta_h(t) = \begin{cases} 0, & 0 \le t < t_1 - h, \\ \frac{1}{h}(t - t_1 + h), & t_1 - h \le t < t_1, \\ 1, & t_1 \le t < t_2, \\ -\frac{1}{h}(t - t_2 - h), & t_2 \le t < t_2 + h, \\ 0, & t_2 + h \le t < T. \end{cases}$$

For small enough $h > 0$, $\varphi_h = \varphi\zeta_h \in \mathrm{Lip}(\Omega_T)$ with $\operatorname{supp}\varphi_h \Subset \Omega \times (0,T)$ and therefore admissible as a test function in the definition of variational solution. Thus

$$-\int_0^T \int_\Omega u(t)\frac{\partial \varphi_h}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_0^T \|Du(t)\|(\Omega)\,\mathrm{d}t \le \int_0^T \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t. \tag{3.2}$$

Notice that

$$\frac{\partial \varphi_h}{\partial t} = \frac{\partial \varphi}{\partial t}\zeta_h + \varphi\zeta_h' = \frac{\partial \varphi}{\partial t}\zeta_h + \begin{cases} 0, & 0 \le t < t_1 - h, \\ \frac{1}{h}\left(\frac{\partial \varphi}{\partial t}(t - t_1 + h) + \varphi\right), & t_1 - h \le t < t_1, \\ \frac{\partial \varphi}{\partial t}, & t_1 \le t < t_2, \\ -\frac{1}{h}\left(\frac{\partial \varphi}{\partial t}(t - t_2 - h) + \varphi\right), & t_2 \le t < t_2 + h, \\ 0, & t_2 + h \le t < T. \end{cases}$$

For the first term on the left-hand side of (3.2), we find

$$\begin{aligned} -\int_0^T \int_\Omega u(t)\frac{\partial \varphi_h}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t &= -\frac{1}{h}\int_{t_1-h}^{t_1} \int_\Omega u(t)\left(\frac{\partial \varphi}{\partial t}(t - t_1 + h) + \varphi(t)\right)\mathrm{d}\mu\,\mathrm{d}t \\ &\quad - \int_{t_1}^{t_2} \int_\Omega u(t)\frac{\partial \varphi}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \frac{1}{h}\int_{t_2}^{t_2+h} \int_\Omega u(t)\left(\frac{\partial \varphi}{\partial t}(t - t_2 - h) + \varphi(t)\right)\mathrm{d}\mu\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{h}\int_{t_1-h}^{t_1}\int_{\Omega} u(t)\frac{\partial\varphi}{\partial t}(t-t_1+h)\,\mathrm{d}\mu\,\mathrm{d}t - \frac{1}{h}\int_{t_1-h}^{t_1}\int_{\Omega} u(t)\varphi(t)\,\mathrm{d}\mu\,\mathrm{d}t \\
&\quad - \int_{t_1}^{t_2}\int_{\Omega} u(t)\frac{\partial\varphi}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \frac{1}{h}\int_{t_2}^{t_2+h}\int_{\Omega} u(t)\frac{\partial\varphi}{\partial t}(t-t_2-h)\,\mathrm{d}\mu\,\mathrm{d}t \\
&\quad + \frac{1}{h}\int_{t_2}^{t_2+h}\int_{\Omega} u(t)\varphi(t)\,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}
$$

Since supp $\varphi \Subset \Omega' \times (0,T)$, we have

$$
-\int_{t_1}^{t_2}\int_{\Omega} u(t)\frac{\partial\varphi}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_{\Omega'} u(t)\frac{\partial\varphi}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t.
$$

By the dominated convergence theorem, we get

$$
-\frac{1}{h}\int_{t_1-h}^{t_1}\int_{\Omega} u(t)\frac{\partial\varphi}{\partial t}(t-t_1+h)\,\mathrm{d}\mu\,\mathrm{d}t \xrightarrow{h\to 0} 0,
$$

and

$$
\frac{1}{h}\int_{t_2}^{t_2+h}\int_{\Omega} u(t)\frac{\partial\varphi}{\partial t}(t-t_2-h)\,\mathrm{d}\mu\,\mathrm{d}t \xrightarrow{h\to 0} 0.
$$

on the other hand, by the Lebesgue differentiation theorem, we obtain

$$
-\frac{1}{h}\int_{t_1-h}^{t_1}\int_{\Omega} u(t)\varphi(t)\,\mathrm{d}\mu\,\mathrm{d}t \xrightarrow{h\to 0} -\int_{\Omega'} u(t_1)\varphi(t_1)\,\mathrm{d}\mu
$$

and

$$
\frac{1}{h}\int_{t_2}^{t_2+h}\int_{\Omega} u(t)\varphi(t)\,\mathrm{d}\mu\,\mathrm{d}t \xrightarrow{h\to 0} \int_{\Omega'} u(t_2)\varphi(t_2)\,\mathrm{d}\mu,
$$

for $\mathcal{L}^1$-almost every $t_1, t_2 \in (0,T)$. This implies that

$$
-\int_0^T\int_{\Omega} u(t)\frac{\partial\varphi_h}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t \xrightarrow{h\to 0} -\int_{t_1}^{t_2}\int_{\Omega'} u(t)\frac{\partial\varphi}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \left[\int_{\Omega'} u(t)\varphi(t)\,\mathrm{d}\mu\right]_{t=t_1}^{t_2}, \tag{3.3}
$$

for $\mathcal{L}^1$-almost every $t_1, t_2 \in (0,T)$.

For the second term on the left-hand side of (3.2), we find

$$
\begin{aligned}
\int_0^T \|Du(t)\|(\Omega)\,\mathrm{d}t &= \int_0^{t_1-h} \|Du(t)\|(\Omega)\,\mathrm{d}t + \int_{t_1-h}^{t_1} \|Du(t)\|(\Omega)\,\mathrm{d}t + \int_{t_1}^{t_2} \|Du(t)\|(\Omega)\,\mathrm{d}t \\
&\quad + \int_{t_2}^{t_2+h} \|Du(t)\|(\Omega)\,\mathrm{d}t + \int_{t_2+h}^{T} \|Du(t)\|(\Omega)\,\mathrm{d}t \\
&= \int_0^{t_1-h} \|Du(t)\|(\Omega)\,\mathrm{d}t + \int_{t_1-h}^{t_1} \|Du(t)\|(\Omega)\,\mathrm{d}t + \int_{t_1}^{t_2} \|Du(t)\|(\Omega\setminus\Omega')\,\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \|Du(t)\|(\Omega')\,\mathrm{d}t + \int_{t_2}^{t_2+h} \|Du(t)\|(\Omega)\,\mathrm{d}t + \int_{t_2+h}^{T} \|Du(t)\|(\Omega)\,\mathrm{d}t.
\end{aligned}
$$

Here

$$
\int_{t_1-h}^{t_1} \|Du(t)\|(\Omega)\,\mathrm{d}t \xrightarrow{h\to 0} 0 \quad \text{and} \quad \int_{t_2}^{t_2+h} \|Du(t)\|(\Omega)\,\mathrm{d}t \xrightarrow{h\to 0} 0.
$$

Moreover,

$$\begin{aligned}\int_0^T \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t &= \int_0^{t_1-h} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t + \int_{t_1-h}^{t_1} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t \\ &\quad + \int_{t_1}^{t_2} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t + \int_{t_2}^{t_2+h} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t \\ &\quad + \int_{t_2+h}^{T} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t \\ &= \int_0^{t_1-h} \|Du(t)\|(\Omega)\,\mathrm{d}t + \int_{t_1-h}^{t_1} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t + \int_{t_1}^{t_2} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t \\ &\quad + \int_{t_2}^{t_2+h} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t + \int_{t_2+h}^{T} \|Du(t)\|(\Omega)\,\mathrm{d}t,\end{aligned}$$

where

$$\int_{t_1-h}^{t_1} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t \xrightarrow{h\to 0} 0, \qquad \int_{t_2}^{t_2+h} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t \xrightarrow{h\to 0} 0,$$

and

$$\begin{aligned}\int_{t_1}^{t_2} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t &\xrightarrow{h\to 0} \int_{t_1}^{t_2} \|D(u+\varphi)(t)\|(\Omega)\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} \|D(u+\varphi)(t)\|(\Omega\setminus\Omega')\,\mathrm{d}t + \int_{t_1}^{t_2} \|D(u+\varphi)(t)\|(\Omega')\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} \|Du(t)\|(\Omega\setminus\Omega')\,\mathrm{d}t + \int_{t_1}^{t_2} \|D(u+\varphi)(t)\|(\Omega')\,\mathrm{d}t.\end{aligned}$$

Here we used the fact that supp $\varphi \Subset \Omega' \times (0,T)$. Substituting these in (3.2), we obtain

$$\begin{aligned}&-\int_{t_1}^{t_2}\int_{\Omega'} u(t)\frac{\partial\varphi}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \left[\int_{\Omega'} u(t)\varphi(t)\,\mathrm{d}\mu\right]_{t=t_1}^{t_2} + \int_0^{t_1-h} \|Du(t)\|(\Omega)\,\mathrm{d}t \\ &\quad + \int_{t_1}^{t_2} \|Du(t)\|(\Omega')\,\mathrm{d}t + \int_{t_2+h}^{T} \|Du(t)\|(\Omega)\,\mathrm{d}t \\ &\le \int_0^{t_1-h} \|Du(t)\|(\Omega)\,\mathrm{d}t + \int_{t_1}^{t_2} \|D(u+\varphi_h)(t)\|(\Omega)\,\mathrm{d}t + \int_{t_2+h}^{T} \|Du(t)\|(\Omega)\,\mathrm{d}t.\end{aligned}$$

Eliminating the repeated elements gives

$$\int_{t_1}^{t_2}\left(\int_{\Omega'} -u(t)\frac{\partial\varphi}{\partial t}(t)\,\mathrm{d}\mu + \|Du(t)\|(\Omega')\right)\mathrm{d}t \le \int_{t_1}^{t_2} \|D(u+\varphi)(t)\|(\Omega')\,\mathrm{d}t - \left[\int_{\Omega'} u(t)\varphi(t)\,\mathrm{d}\mu\right]_{t=t_1}^{t_2}. \quad \square$$

4. Parabolic De Giorgi class

Next we define the class of functions for which we prove the regularity results. For $(x_0,t_0)\in X\times\mathbb{R}$ and $\rho,\theta>0$, we denote $Q^-_{\rho,\theta}(x_0,t_0) = B_\rho(x_0)\times(t_0-\theta\rho, t_0]$. The positive and negative parts of u are denoted by $u_\pm = \max\{\pm u, 0\}$, respectively.

Definition 4.1. A function $u \in L^1_{\mathrm{loc}}(0,T;BV_{\mathrm{loc}}(\Omega))$ belongs to the parabolic De Giorgi class $DG^{\pm}(\Omega_T;\gamma)$, with $\gamma > 0$, if

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_0-\theta\rho\le t\le t_0} \int_{B_\rho(x_0)} \varphi(t)(u(t)-k)^2_{\pm}\,\mathrm{d}\mu + \int_{t_0-\theta\rho}^{t_0} \|D(\varphi(u-k)_+)(t)\|(B_\rho(x_0))\,\mathrm{d}t \\
&\quad\le \gamma \iint_{Q^-_{\rho,\theta}(x_0,t_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right| (u(t)-k)^2_{\pm}\,\mathrm{d}\mu\,\mathrm{d}t + \gamma \iint_{Q^-_{\rho,\theta}(x_0,t_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t \\
&\qquad - \left[\int_{B_\rho(x_0)} \varphi(t)(u(t)-k)^2_{\pm}\,\mathrm{d}\mu\right]_{t=t_0-\theta\rho}^{t_0},
\end{aligned} \tag{4.1}$$

for every $Q^-_{\rho,\theta}(x_0,t_0) \Subset \Omega_T$, $k \in \mathbb{R}$ and $\varphi \in \mathrm{Lip}(\Omega_T)$ with $\operatorname{supp}\varphi \Subset B_\rho(x_0)\times(0,T)$ and $0 \le \varphi \le 1$.

The parabolic De Giorgi class $DG(\Omega_T;\gamma)$ is defined as

$$DG(\Omega_T;\gamma) = DG^+(\Omega_T;\gamma) \cap DG^-(\Omega_T;\gamma).$$

The proof of the necessary and sufficient conditions for continuity of a variational solution to the total variation flow, Theorem 7.1, will only use the local integral inequalities in (4.1). We show that a variational solution to the total variation flow belongs to the parabolic De Giorgi class.

Proposition 4.2. *Let u be variational solution to the total variation flow in Ω_T. Then $u \in DG(\Omega_T;8)$.*

Proof. Let $\phi \in \mathrm{Lip}(\Omega_T)$ with $\operatorname{supp}\phi \Subset \Omega_T$. There exists $h_0 > 0$ such that for every $0 < h < h_0$, we have $\phi_h \in \mathrm{Lip}(\Omega_T)$ with $\operatorname{supp}\phi_h \Subset \Omega_T$ and thus we may apply it as test function in (3.1). Here ϕ_h denotes the time mollification of ϕ. For a small enough s the translated function $v(t) = u(t-s)$ fulfills (3.1). For $0 < t_2 < t_1 < T$, to be specified later, Proposition 3.2 implies

$$\begin{aligned}
&-\int_{t_2}^{t_1}\int_{B_\rho(x_0)} v(t)\frac{\partial\phi_h}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1} \|Dv(t)\|(B_\rho(x_0))\,\mathrm{d}t \\
&\quad\le \int_{t_2}^{t_1} \|D(v+\phi_h)(t)\|(B_\rho(x_0))\,\mathrm{d}t - \left[\int_{B_\rho(x_0)} v(t)\phi_h(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1}.
\end{aligned}$$

Let $(u_i)_{i\in\mathbb{N}}$ be a minimizing sequence with $u_i \in L^1_{\mathrm{loc}}(0,T;N^{1,1}_{\mathrm{loc}}(\Omega))$ for every $i \in \mathbb{N}$, $u_i \to u$ in $L^1_{\mathrm{loc}}(0,T;N^{1,1}_{\mathrm{loc}}(\Omega))$ as $i \to \infty$ and

$$\begin{aligned}
\int_{t_2}^{t_1} \|Dv(t)\|(B_\rho(x_0))\,\mathrm{d}t &= \int_{t_2}^{t_1} \|Du(t-s)\|(B_\rho(x_0))\,\mathrm{d}t \\
&= \lim_{i\to\infty} \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{u_i}(t-s)\,\mathrm{d}\mu\,\mathrm{d}t \\
&= \lim_{i\to\infty} \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t,
\end{aligned}$$

where $v_i(t) = u_i(t-s)$ for every $i \in \mathbb{N}$. Let $\epsilon > 0$. There exists $i_\epsilon \in \mathbb{N}$ such that, for every $i \ge i_\epsilon$, we have

$$\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t \le \int_{t_2}^{t_1} \|Dv(t)\|(B_\rho(x_0))\,\mathrm{d}t + \frac{\epsilon}{2},$$

and

$$\int_{t_2}^{t_1} \|D(v+\phi_h)(t)\|(B_\rho(x_0))\,\mathrm{d}t \le \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i+\phi_h}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \frac{\epsilon}{2}.$$

This implies

$$
\begin{aligned}
-\int_{t_2}^{t_1}\int_{B_\rho(x_0)} v(t)\frac{\partial\phi_h}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\
\le \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i+\phi_h}(t)\,\mathrm{d}\mu\,\mathrm{d}t - \left[\int_{B_\rho(x_0)} v(t)\phi_h(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1} + \epsilon \\
\le \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i+\phi}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{\phi_h-\phi}(t)\,\mathrm{d}\mu\,\mathrm{d}t - \left[\int_{B_\rho(x_0)} v(t)\phi_h(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1} + \epsilon,
\end{aligned}
$$

for every $i \ge i_\epsilon$.

Let $i \ge i_\epsilon$. We multiply both sides of the inequality above by a standard mollifier $\eta_\varepsilon = \eta_\varepsilon(s)$ with support $[-\varepsilon, \varepsilon]$ for small enough $\varepsilon > 0$. By integrating the resulting expression in the variable s we obtain

$$
\begin{aligned}
-\int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} v(t)\eta_\varepsilon(s)\frac{\partial\phi_h}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s + \int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s \\
\le \int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i+\phi}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s + \int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{\phi_h-\phi}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s \\
- \left[\int_{-\varepsilon}^{\varepsilon}\int_{B_\rho(x_0)} v(t)\eta_\varepsilon(s)\phi_h(t)\,\mathrm{d}\mu\,\mathrm{d}s\right]_{t=t_2}^{t_1} + \epsilon.
\end{aligned}
$$

Applying integration by parts and Fubini's theorem, we have

$$
\begin{aligned}
\int_{t_2}^{t_1}\int_{B_\rho(x_0)} \frac{\partial u_\varepsilon}{\partial t}(t)\phi_h(t)\,\mathrm{d}\mu\,\mathrm{d}t - \left[\int_{B_\rho(x_0)} u_\varepsilon(t)\phi_h(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1} + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} (g_{u_i})_\varepsilon(t)\,\mathrm{d}\mu\,\mathrm{d}t \\
= -\int_{t_2}^{t_1}\int_{B_\rho(x_0)} u_\varepsilon(t)\frac{\partial\phi_h}{\partial t}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} (g_{u_i})_\varepsilon(t)\,\mathrm{d}\mu\,\mathrm{d}t \\
\le \int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i+\phi}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{\phi_h-\phi}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\
- \left[\int_{B_\rho(x_0)} u_\varepsilon(t)\phi_h(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1} + \epsilon.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{t_2}^{t_1}\int_{B_\rho(x_0)} \frac{\partial u_\varepsilon}{\partial t}(t)\phi_h(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} (g_{u_i})_\varepsilon(t)\,\mathrm{d}\mu\,\mathrm{d}t \\
\le \int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i+\phi}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{\phi_h-\phi}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \epsilon.
\end{aligned}
$$

Lemma 2.3 implies that the last term on the right-hand side converges to zero as $h \to 0$. By passing to the limit $h \to 0$, we have

$$
\begin{aligned}
\int_{t_2}^{t_1}\int_{B_\rho(x_0)} \frac{\partial u_\varepsilon}{\partial t}(t)\phi(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} (g_{u_i})_\varepsilon(t)\,\mathrm{d}\mu\,\mathrm{d}t \\
\le \int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i+\phi}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s + \epsilon.
\end{aligned} \tag{4.2}
$$

Let $\varphi \in \mathrm{Lip}(\Omega_T)$ with $\operatorname{supp}\varphi \Subset B_\rho(x_0) \times (0,T)$ and $0 \le \varphi \le 1$. Let ζ_h be a cutoff function depending only on time defined as

$$\zeta_h(t) = \begin{cases} \frac{1}{h}(t - t_0 + \theta\rho + h), & t_0 - \theta\rho - h \le t < t_0 - \theta\rho, \\ 1, & t_0 - \theta\rho \le t \le t_0, \\ -\frac{1}{h}(t - t_0 - h), & t_0 < t \le t_0 + h, \\ 0, & \text{otherwise.} \end{cases}$$

We apply the test function $\phi = -\varphi\zeta_h((u_i)_\varepsilon - k)_+$ in (4.2). For the right-hand side of (4.2) we have

$$\begin{aligned} &\int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i - \varphi\zeta_h((u_i)_\varepsilon - k)_+}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s \\ &\quad \le \int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i-u_i}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s + \int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{u_i-(u_i)_\varepsilon}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s \\ &\qquad + \int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{(u_i)_\varepsilon - \varphi\zeta_h((u_i)_\varepsilon - k)_+}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s \\ &\quad = \int_{-\varepsilon}^{\varepsilon}\left(\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i-u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t\right)\eta_\varepsilon(s)\,\mathrm{d}s + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{u_i-(u_i)_\varepsilon}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\qquad + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{(u_i)_\varepsilon - \varphi\zeta_h((u_i)_\varepsilon - k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned} \tag{4.3}$$

Denote

$$A_\varepsilon(t) = \{x \in B_\rho(x_0) : (u_i)_\varepsilon(x,t) > k\}.$$

For the last integral in (4.3), we have

$$\begin{aligned} &\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{(u_i)_\varepsilon - \varphi\zeta_h((u_i)_\varepsilon - k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad = \int_{t_2}^{t_1}\int_{A_\varepsilon(t)} g_{(u_i)_\varepsilon - \varphi\zeta_h((u_i)_\varepsilon - k)}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)\setminus A_\varepsilon(t)} g_{(u_i)_\varepsilon}(t)\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned}$$

The Leibniz rule for upper gradients implies

$$\begin{aligned} &\int_{t_2}^{t_1}\int_{A_\varepsilon(t)} g_{(u_i)_\varepsilon - \varphi\zeta_h((u_i)_\varepsilon - k)}(t)\,\mathrm{d}\mu\,\mathrm{d}t = \int_{t_2}^{t_1}\int_{A_\varepsilon(t)} g_{(1-\varphi\zeta_h)((u_i)_\varepsilon - k)}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad \le \int_{t_2}^{t_1}\int_{A_\varepsilon(t)} (1 - \varphi(t)\zeta_h(t))g_{(u_i)_\varepsilon - k}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{A_\varepsilon(t)} ((u_i)_\varepsilon(t) - k)g_{\varphi\zeta_h}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad = \int_{t_2}^{t_1}\int_{B_\rho(x_0)} (1 - \varphi(t)\zeta_h(t))g_{((u_i)_\varepsilon - k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} ((u_i)_\varepsilon(t) - k)_+ g_{\varphi\zeta_h}(t)\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned}$$

For the integral on the right-hand side of (4.2), we have

$$\begin{aligned} &\int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i - \varphi\zeta_h((u_i)_\varepsilon - k)_+}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s \\ &\le \int_{-\varepsilon}^{\varepsilon}\left(\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i-u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t\right)\eta_\varepsilon(s)\,\mathrm{d}s + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{u_i-(u_i)_\varepsilon}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} (1 - \varphi(t)\zeta_h(t))g_{((u_i)_\varepsilon - k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} ((u_i)_\varepsilon(t) - k)_+ g_{\varphi\zeta_h}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad + \int_{t_2}^{t_1}\int_{B_\rho(x_0)\setminus A_\varepsilon(t)} g_{(u_i)_\varepsilon}(t)\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned}$$

By letting $h \to 0$, we obtain

$$\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i-\varphi\chi_{[t_0-\theta\rho,t_0]}((u_i)_\varepsilon-k)_+}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s\\
&\quad\le \int_{-\varepsilon}^{\varepsilon}\left(\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i-u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t\right)\eta_\varepsilon(s)\,\mathrm{d}s + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{u_i-(u_i)_\varepsilon}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} (1-\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t))g_{((u_i)_\varepsilon-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} \chi_{[t_0-\theta\rho,t_0]}(t)((u_i)_\varepsilon(t)-k)_+ g_\varphi(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)\setminus A_\varepsilon(t)} g_{(u_i)_\varepsilon}(t)\,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}$$

Lemma 2.3 implies that

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\left(\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i-u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t\right)\eta_\varepsilon(s)\,\mathrm{d}s = 0,$$

and

$$\lim_{\varepsilon\to 0}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{u_i-(u_i)_\varepsilon}(t)\,\mathrm{d}\mu\,\mathrm{d}t = 0.$$

By letting $\varepsilon \to 0$, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\int_{t_2}^{t_1}\int_{B_\rho(x_0)} g_{v_i-\varphi\chi_{[t_0-\theta\rho,t_0]}((u_i)_\varepsilon-k)_+}(t)\eta_\varepsilon(s)\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}s\\
&\quad\le \int_{t_2}^{t_1}\int_{B_\rho(x_0)} (1-\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t))g_{(u_i-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad + \int_{t_2}^{t_1}\int_{B_\rho(x_0)} \chi_{[t_0-\theta\rho,t_0]}(t)(u_i(t)-k)_+ g_\varphi(t)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{t_2}^{t_1}\int_{B_\rho(x_0)\setminus A(t)} g_{u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t,
\end{aligned}$$

where

$$A(t) = \{x \in B_\rho(x_0) : u_i(x,t) > k\}.$$

Here we used the following observation

$$\int_{-\infty}^{\infty} |\chi_{A_\varepsilon(t)}(x) - \chi_{A(t)}(x)|\,\mathrm{d}k = \int_{\min\{(u_i)_\varepsilon(x,t),u_i(x,t)\}}^{\max\{(u_i)_\varepsilon(x,t),u_i(x,t)\}} 1\,\mathrm{d}k = |(u_i)_\varepsilon(x,t) - u_i(x,t)|.$$

By Fubini's Theorem, we get

$$\int_{-\infty}^{\infty}\int_0^T\int_{B_\rho(x_0)} |\chi_{A_\varepsilon(t)} - \chi_{A(t)}|\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}k = \int_0^T\int_{B_\rho(x_0)} |(u_i)_\varepsilon(t) - u_i(t)|\,\mathrm{d}\mu\,\mathrm{d}t.$$

By Lemma 2.3, $(u_i)_\varepsilon \to u_i$ in $L^1_{\mathrm{loc}}(0,T,N^{1,1}_{\mathrm{loc}}(\Omega))$, in particular we have $(u_i)_\varepsilon \to u_i$ in $L^1_{\mathrm{loc}}(0,T,L^1_{\mathrm{loc}}(\Omega))$ as $\varepsilon \to 0$. Therefore,

$$0 = \lim_{\varepsilon\to 0}\int_0^T\int_{B_\rho(x_0)} |(u_i)_\varepsilon(t) - u_i(t)|\,\mathrm{d}\mu\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\int_0^T\int_{B_\rho(x_0)} |\chi_{A_\varepsilon(t)} - \chi_{A(t)}|\,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}k.$$

By Fatou's lemma, we have

$$0 \le \int_{-\infty}^{\infty} \left(\lim_{\varepsilon\to 0} \int_0^T \int_{B_\rho(x_0)} |\chi_{A_\varepsilon(t)} - \chi_{A(t)}| \,\mathrm{d}\mu\,\mathrm{d}t \right) \mathrm{d}k$$
$$\le \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \int_0^T \int_{B_\rho(x_0)} |\chi_{A_\varepsilon(t)} - \chi_{A(t)}| \,\mathrm{d}\mu\,\mathrm{d}t\,\mathrm{d}k = 0.$$

This implies that

$$\lim_{\varepsilon\to 0} \int_0^T \int_{B_\rho(x_0)} |\chi_{A_\varepsilon(t)} - \chi_{A(t)}| \,\mathrm{d}\mu\,\mathrm{d}t = 0, \tag{4.4}$$

for $\mathcal{L}^1$-almost every k. Let $k \in \mathbb{R}$ be such that (4.4) holds. Applying Fatou's lemma one more time, we get

$$0 \le \int_0^T \left(\lim_{\varepsilon\to 0} \int_{B_\rho(x_0)} |\chi_{A_\varepsilon(t)} - \chi_{A(t)}| \,\mathrm{d}\mu \right) \mathrm{d}t \le \lim_{\varepsilon\to 0} \int_0^T \int_{B_\rho(x_0)} |\chi_{A_\varepsilon(t)} - \chi_{A(t)}| \,\mathrm{d}\mu\,\mathrm{d}t = 0.$$

It follows that

$$\lim_{\varepsilon\to 0} \int_{B_\rho(x_0)} |\chi_{A_\varepsilon(t)} - \chi_{A(t)}| \,\mathrm{d}\mu = 0,$$

for $\mathcal{L}^1$-almost every $t \in (0,T)$. Therefore, we conclude $\chi_{A_\varepsilon(t)} \to \chi_{A(t)}$ in $L^1(B_\rho(x_0))$ as $\varepsilon \to 0$, for $\mathcal{L}^1$-almost every $t \in (0,T)$ and for $\mathcal{L}^1$-almost every $k \in \mathbb{R}$.

For the first term on the left-hand side of (4.2) we find

$$\begin{aligned}\int_{t_2}^{t_1} \int_{B_\rho(x_0)} \frac{\partial u_\varepsilon}{\partial t}(t)\phi(t)\,\mathrm{d}\mu\,\mathrm{d}t &= -\int_{t_2}^{t_1} \int_{B_\rho(x_0)} \frac{\partial u_\varepsilon}{\partial t}(t)(u_\varepsilon(t)-k)_+\varphi(t)\zeta_h(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\xrightarrow{h\to 0} -\int_{t_2}^{t_1} \int_{B_\rho(x_0)} \frac{\partial u_\varepsilon}{\partial t}(t)(u_\varepsilon(t)-k)_+\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\,\mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}&-\int_{t_2}^{t_1} \int_{B_\rho(x_0)} \frac{\partial u_\varepsilon}{\partial t}(t)(u_\varepsilon(t)-k)_+\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad= -\int_{t_2}^{t_1} \int_{B_\rho(x_0)} \frac{\partial u_\varepsilon}{\partial t}(t)(u_\varepsilon(t)-k)_+\varphi_1(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad= \frac{1}{2}\int_{t_2}^{t_1} \int_{B_\rho(x_0)} \frac{\partial}{\partial t}\left((u_\varepsilon-k)_+^2\varphi_1\right)(t)\,\mathrm{d}\mu\,\mathrm{d}t - \frac{1}{2}\int_{t_2}^{t_1} \int_{B_\rho(x_0)} \frac{\partial \varphi_1}{\partial t}(t)(u_\varepsilon(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad= \frac{1}{2}\left[\int_{B_\rho(x_0)} (u_\varepsilon(t)-k)_+^2\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1} \\ &\qquad - \frac{1}{2}\int_{t_2}^{t_1} \int_{B_\rho(x_0)} \frac{\partial \varphi}{\partial t}(t)(u_\varepsilon(t)-k)_+^2\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad\xrightarrow{\varepsilon\to 0} \frac{1}{2}\left[\int_{B_\rho(x_0)} (u(t)-k)_+^2\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1} \\ &\qquad - \frac{1}{2}\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}} \int_{B_\rho(x_0)} \frac{\partial \varphi}{\partial t}(t)(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t.\end{aligned}$$

For the second term on the left-hand side of (4.2), by Lemma 2.3, we have

$$\lim_{\varepsilon\to 0} \int_{t_2}^{t_1} \int_{B_\rho(x_0)} (g_{u_i})_\varepsilon(t)\,\mathrm{d}\mu\,\mathrm{d}t = \int_{t_2}^{t_1} \int_{B_\rho(x_0)} g_{u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t.$$

Thus, by (4.2) we get

$$\begin{aligned}
&\frac{1}{2}\left[\int_{B_\rho(x_0)}(u(t)-k)_+^2\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1}+\int_{t_2}^{t_1}\int_{B_\rho(x_0)}g_{u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\quad\le\frac{1}{2}\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}\left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad+\int_{t_2}^{t_1}\int_{B_\rho(x_0)}(1-\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t))g_{(u_i-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad+\int_{t_2}^{t_1}\int_{B_\rho(x_0)}(u_i(t)-k)_+g_\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\,\mathrm{d}t+\int_{t_2}^{t_1}\int_{B_\rho(x_0)\setminus A(t)}g_{u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t,
\end{aligned}$$

and consequently

$$\begin{aligned}
&\frac{1}{2}\left[\int_{B_\rho(x_0)}(u(t)-k)_+^2\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1}+\int_{t_2}^{t_1}\int_{A(t)}g_{u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\quad\le\frac{1}{2}\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}\left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad+\int_{t_2}^{t_1}\int_{B_\rho(x_0)}(1-\varphi(t)\chi_{[t_2,t_1]}(t))g_{(u_i-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad+\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}(u_i(t)-k)_+g_\varphi(t)\,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}$$

Since

$$\int_{t_2}^{t_1}\int_{A(t)}g_{u_i}(t)\,\mathrm{d}\mu\,\mathrm{d}t=\int_{t_2}^{t_1}\int_{B_\rho(x_0)}g_{(u_i-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t,$$

by absorbing terms, this implies

$$\begin{aligned}
&\left[\frac{1}{2}\int_{B_\rho(x_0)}(u(t)-k)_+^2\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1}+\int_{t_2}^{t_1}\int_{B_\rho(x_0)}\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)g_{(u_i-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\quad\le\frac{1}{2}\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}\left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad+\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}g_\varphi(t)(u_i(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}$$

By the Leibniz rule we obtain

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{B_\rho(x_0)}\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)g_{(u_i-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t=\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}\varphi(t)g_{(u_i-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\quad\ge\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}g_{\varphi(u_i-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t-\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}g_\varphi(t)(u_i(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\frac{1}{2}\left[\int_{B_\rho(x_0)}(u(t)-k)_+^2\varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1}+\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}g_{\varphi(u_i-k)_+}(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\quad\le\frac{1}{2}\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}\left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad+2\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)}g_\varphi(t)(u_i(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}$$

Letting $i \to \infty$, we obtain

$$\begin{aligned} &\frac{1}{2}\left[\int_{B_\rho(x_0)} (u(t)-k)_+^2 \varphi(t)\chi_{[t_0-\theta\rho,t_0]}(t)\,\mathrm{d}\mu\right]_{t=t_2}^{t_1} + \int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}} \|D(\varphi(u-k)_+)(t)\|(B_\rho(x_0))\,\mathrm{d}t \\ &\quad\le \frac{1}{2}\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right| (u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t \\ &\qquad + 2\int_{\max\{t_2,t_0-\theta\rho\}}^{\min\{t_1,t_0\}}\int_{B_\rho(x_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned} \tag{4.5}$$

Choosing $t_2 \in (0,T)$, with $t_2 < t_0-\theta\rho$ then, for all $t_0-\theta\rho \le t_1 \le t_0$, by (4.5), we have

$$\begin{aligned} \frac{1}{2}\int_{B_\rho(x_0)} (u(t_1)-k)_+^2\varphi(t_1)\,\mathrm{d}\mu &\le \frac{1}{2}\int_{B_\rho(x_0)} (u(t_1)-k)_+^2\varphi(t_1)\,\mathrm{d}\mu \\ &\quad + \int_{t_0-\theta\rho}^{t_1} \|D(\varphi(u-k)_+)(t)\|(B_\rho(x_0))\,\mathrm{d}t \\ &\le \frac{1}{2}\int_{t_0-\theta\rho}^{t_1}\int_{B_\rho(x_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t + 2\int_{t_0-\theta\rho}^{t_1}\int_{B_\rho(x_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_{B_\rho(x_0)} (u(t_2)-k)_+^2\varphi(t_2)\chi_{[t_0-\theta\rho,t_0]}(t_2)\,\mathrm{d}\mu \\ &= \frac{1}{2}\int_{t_0-\theta\rho}^{t_1}\int_{B_\rho(x_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t + 2\int_{t_0-\theta\rho}^{t_1}\int_{B_\rho(x_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t \\ &\le \frac{1}{2}\int_{t_0-\theta\rho}^{t_0}\int_{B_\rho(x_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t + 2\int_{t_0-\theta\rho}^{t_0}\int_{B_\rho(x_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned}$$

We conclude that

$$\begin{aligned} \frac{1}{2}\int_{B_\rho(x_0)} (u(t_1)-k)_+^2\varphi(t_1)\,\mathrm{d}\mu &\le \frac{1}{2}\int_{t_0-\theta\rho}^{t_0}\int_{B_\rho(x_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad + 2\int_{t_0-\theta\rho}^{t_0}\int_{B_\rho(x_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t, \end{aligned}$$

and this holds for any $t_0-\theta\rho \le t_1 \le t_0$. Therefore

$$\begin{aligned} \operatorname*{ess\,sup}_{t_0-\theta\rho\le t\le t_0} \int_{B_\rho(x_0)} (u(t)-k)_+^2\varphi(t)\,\mathrm{d}\mu &\le \iint_{Q^-_{\rho,\theta}(x_0,t_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad + 4\iint_{Q^-_{\rho,\theta}(x_0,t_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned} \tag{4.6}$$

On the other hand, by choosing $t_2 = t_0-\theta\rho$ and $t_1 = t_0$ in (4.5), we have

$$\begin{aligned} \frac{1}{2}\int_{t_0-\theta\rho}^{t_0} \|D(\varphi(u-k)_+)(t)\|(B_\rho(x_0))\,\mathrm{d}t &\le \int_{t_0-\theta\rho}^{t_0} \|D(\varphi(u-k)_+)(t)\|(B_\rho(x_0))\,\mathrm{d}t \\ &\le \frac{1}{2}\iint_{Q^-_{\rho,\theta}(x_0,t_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right|(u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t + 2\iint_{Q^-_{\rho,\theta}(x_0,t_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t \\ &\quad - \left[\frac{1}{2}\int_{B_\rho(x_0)} (u(t)-k)_+^2\varphi(t)\,\mathrm{d}\mu\right]_{t=t_0-\theta\rho}^{t_0}. \end{aligned}$$

This implies that

$$\int_{t_0-\theta\rho}^{t_0} \|D(\varphi(u-k)_+)(t)\|(B_\rho(x_0))\,\mathrm{d}t \le \iint_{Q^-_{\rho,\theta}(x_0,t_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right| (u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t + 4\iint_{Q^-_{\rho,\theta}(x_0,t_0)} g_\varphi(t)(u(t)-k)_+\,\mathrm{d}\mu\,\mathrm{d}t - \left[\int_{B_\rho(x_0)} (u(t)-k)_+^2\varphi(t)\,\mathrm{d}\mu\right]_{t=t_0-\theta\rho}^{t_0}. \tag{4.7}$$

Adding (4.6) and (4.7) we conclude that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_0-\theta\rho\le t\le t_0} \int_{B_\rho(x_0)} (u(t)-k)_+^2\varphi(t)\,\mathrm{d}\mu + \int_{t_0-\theta\rho}^{t_0} \|D(\varphi(u-k)_+)(t)\|(B_\rho(x_0))\,\mathrm{d}t\\
&\le 2\iint_{Q^-_{\rho,\theta}(x_0,t_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right| (u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t + 8\iint_{Q^-_{\rho,\theta}(x_0,t_0)} (u(t)-k)_+ g_\varphi(t)\,\mathrm{d}\mu\,\mathrm{d}t\\
&\quad - \left[\int_{B_\rho(x_0)} \varphi(t)(u(t)-k)_+^2\,\mathrm{d}\mu\right]_{t=t_0-\theta\rho}^{t_0}\\
&\le 8\left(\iint_{Q^-_{\rho,\theta}(x_0,t_0)} \left|\frac{\partial\varphi}{\partial t}(t)\right| (u(t)-k)_+^2\,\mathrm{d}\mu\,\mathrm{d}t + \iint_{Q^-_{\rho,\theta}(x_0,t_0)} (u(t)-k)_+ g_\varphi(t)\,\mathrm{d}\mu\,\mathrm{d}t\right)\\
&\quad - \left[\int_{B_\rho(x_0)} \varphi(t)(u(t)-k)_+^2\,\mathrm{d}\mu\right]_{t=t_0-\theta\rho}^{t_0}.
\end{aligned}$$

This implies $u \in DG^+(\Omega_T;8)$. A similar argument shows that $u \in DG^-(\Omega_T;8)$ and thus $u \in DG(\Omega_T;8)$ □

5. De Giorgi lemma

This short section is devoted to prove that functions in a parabolic De Giorgi class are bounded from below. We apply the following standard iteration lemma in the proof, see [16, Lemma 5.1].

Lemma 5.1. *Let $(Y_n)_{n\in\mathbb{N}_0}$ be a sequence of positive numbers that satisfies*

$$Y_{n+1} \le Cb^n Y_n^{1+\alpha}, \tag{5.1}$$

where $C, b > 1$ and $\alpha > 0$ are given numbers. If $Y_0 \le C^{-1/\alpha}b^{-1/\alpha^2}$, then $Y_n \to 0$ as $n\to\infty$.

Let $\rho,\theta > 0$ be such that $Q^-_{\rho,\theta}(x_0,t_0) \subset \Omega_T$ and let

$$m_+ \ge \operatorname*{ess\,sup}_{Q^-_{\rho,\theta}(x_0,t_0)} u, \quad m_- \le \operatorname*{ess\,inf}_{Q^-_{\rho,\theta}(x_0,t_0)} u \quad \text{and} \quad \omega \ge m_+ - m_-.$$

The following lemma is a version of [15, Lemma 6.1] on metric measure spaces.

Lemma 5.2. *Assume that $u \in DG^-(\Omega_T;\gamma)$.*

(i) *For $a,\xi \in (0,1)$ and $\overline{\theta} \in (0,\theta)$, there exists a constant $\nu_- = \nu_-(\gamma, C_\mu, C_P, \omega, \xi, a, \theta, \overline{\theta})$ such that if*

$$(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\theta}(x_0,t_0) \cap \{u \le m_- + \xi\omega\}) \le \nu_-(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\theta}(x_0,t_0)),$$

then $u \ge m_- + a\xi\omega$ $(\mu\otimes\mathcal{L}^1)$-almost everywhere in $B_{\frac{\rho}{2}}(x_0) \times (t_0 - \overline{\theta}\rho, t_0]$.

(ii) *For* $a, \xi \in (0,1)$ *and* $\overline{\theta} \in (0,\theta)$, *there exists a constant* $\nu_+ = \nu_+(\gamma, C_\mu, C_P, \omega, \xi, a, \theta, \overline{\theta})$ *such that if*

$$(\mu \otimes \mathcal{L}^1)(Q^-_{\rho,\theta}(x_0,t_0) \cap \{u \geq m_+ - \xi\omega\}) \leq \nu_+ (\mu \otimes \mathcal{L}^1)(Q^-_{\rho,\theta}(x_0,t_0)),$$

then $u \leq m_+ - a\xi\omega$ $(\mu \otimes \mathcal{L}^1)$*-almost everywhere in* $B_{\frac{\rho}{2}}(x_0) \times (t_0 - \overline{\theta}\rho, t_0]$.

Proof. We prove (i) and the proof for (ii) is similar. For $n \in \mathbb{N}_0$, let

$$\rho_n = \frac{\rho}{2}\left(1 + \frac{1}{2^n}\right), \quad \theta_n = \overline{\theta} + \frac{1}{2^n}(\theta - \overline{\theta}) \quad \text{and} \quad t_n = t_0 - \theta_n \rho.$$

Then $\rho_n \searrow \frac{\rho}{2}$, $\theta_n \searrow \overline{\theta}$ and $t_n = \nearrow t_0 - \theta\rho$ as $n \to \infty$. Denote $B_n = B_{\rho_n}(x_0)$ and $Q^-_n = B_n \times (t_n, t_0]$. Consider Lipschitz continuous functions ζ_n, $n \in \mathbb{N}$, with $\zeta_n = 1$ in Q^-_{n+1}, $\zeta_n = 0$ in $Q^-_{\rho,\theta}(x_0,t_0) \setminus Q^-_n$,

$$g_{\zeta_n} \leq \frac{1}{\rho_n - \rho_{n+1}} = \frac{2^{n+2}}{\rho} \quad \text{and} \quad 0 \leq (\zeta_n)_t \leq \frac{2^{n+1}}{\theta - \overline{\theta}}\frac{1}{\rho}.$$

For $n \in \mathbb{N}$, let

$$\xi_n = a\xi + \frac{1-a}{2^n}\xi \quad \text{and} \quad k_n = m_- + \xi_n\omega.$$

Then $\xi_n \searrow a\xi$ and $k_n \searrow m_- + a\xi\omega$ as $n \to \infty$.

Denote $A_n = Q^-_n \cap \{u \leq k_n\}$, $n \in \mathbb{N}$. By (4.1) we have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_n<t<t_0} \int_{B_n} \zeta_n(t)(u(t)-k_n)^2_- \,\mathrm{d}\mu + \int_{t_n}^{t_0} \|D(\zeta_n(u-k_n)_-)(t)\|(B_n)\,\mathrm{d}t \\
&\quad \leq \gamma \iint_{Q_n} \left(g_{\zeta_n}(t)(u(t)-k_n)_- + |(\zeta_n)_t(t)|(u(t)-k_n)^2_-\right) \mathrm{d}\mu\,\mathrm{d}t \\
&\quad \leq \gamma \left(\frac{2^{n+2}}{\rho} \iint_{Q^-_n} (u(t)-k_n)_- \,\mathrm{d}\mu\,\mathrm{d}t + \frac{2^{n+1}}{(\theta-\overline{\theta})\rho} \iint_{Q^-_n} (u(t)-k_n)^2_- \,\mathrm{d}\mu\,\mathrm{d}t\right).
\end{aligned}$$

In $\{u \leq k_n\}$ we have

$$\begin{gathered}
0 \leq k_n - u = m_- + \xi_n\omega - u = (m_- - u) + \xi_n\omega \leq \xi_n\omega \leq a\xi\omega, \\
0 \leq (u-k_n)_- \leq a\xi\omega \quad \text{and} \quad (u-k_n)^2_- \leq a^2\xi^2\omega^2.
\end{gathered}$$

It follows that

$$\begin{aligned}
&\frac{2^{n+2}}{\rho} \iint_{Q^-_n} (u(t)-k_n)_- \,\mathrm{d}\mu\,\mathrm{d}t + \frac{2^{n+1}}{(\theta-\overline{\theta})\rho} \iint_{Q^-_n} (u(t)-k_n)^2_- \,\mathrm{d}\mu\,\mathrm{d}t \\
&\quad \leq \frac{2^{n+2}}{\rho} a\xi\omega(\mu \otimes \mathcal{L}^1)(Q^-_n \cap \{u \leq k_n\}) + \frac{2^{n+1}}{(\theta-\overline{\theta})\rho} a^2\xi^2\omega^2(\mu \otimes \mathcal{L}^1)(Q_n \cap \{u \leq k_n\}) \\
&\quad = (\mu \otimes \mathcal{L}^1)(A_n)\left(\frac{2^{n+2}}{\rho} a\xi\omega + \frac{2^{n+1}}{(\theta-\overline{\theta})\rho} a^2\xi^2\omega^2\right).
\end{aligned}$$

This implies

$$\operatorname*{ess\,sup}_{t_n<t<t_0} \int_{B_n} \zeta_n(t)(u(t)-k_n)^2_- \,\mathrm{d}\mu + \int_{t_n}^{t_0} \|D(\zeta_n(u-k_n)_-)(t)\|(B_n)\,\mathrm{d}t \leq 2^n\gamma_1\rho^{-1}(\mu \otimes \mathcal{L}^1)(A_n),$$

where

$$\gamma_1 = 2\gamma a\xi\omega \frac{2(\theta-\overline{\theta}) + a\xi\omega}{(\theta-\overline{\theta})}.$$

By Proposition 2.8 there exists a constant $C = C(C_\mu, C_P)$ such that, for $\kappa = \frac{Q+2}{Q}$, we have

$$\begin{aligned}
&\iint_{Q_n^-} (\zeta_n(t)(u(t)-k_n)_-)^\kappa \,\mathrm{d}\mu\,\mathrm{d}t \\
&\quad\le \frac{C\rho_n}{\mu(B_n)^{\frac{1}{Q}}} \int_{t_n}^{t_0} \|D(\zeta_n(u-k_n)_-)(t)\|\,\mathrm{d}t \left(\operatorname*{ess\,sup}_{t_n<t<t_0} \int_{B_n} \zeta_n(t)(u(t)-k_n)_-^2 \,\mathrm{d}\mu \right)^{\frac{1}{Q}} \\
&\quad\le \frac{C\rho_n}{\mu(B_n)^{\frac{1}{Q}}} \left(\operatorname*{ess\,sup}_{t_n<t<t_0} \int_{B_n} (u(t)-k_n)_-^2 \zeta_n(t)\,\mathrm{d}\mu + \int_{t_n}^{t_0} \|D((u-k_n)_-\zeta_n)(t)\|\,\mathrm{d}t \right)^{1+\frac{1}{Q}} \\
&\quad\le \frac{C\rho_n}{\mu(B_n)^{\frac{1}{Q}}} \left(2^n \gamma_1 \rho^{-1} (\mu\otimes\mathcal{L}^1)(A_n)\right)^{1+\frac{1}{Q}} \\
&\quad\le 2^{n\frac{Q+1}{Q}} \rho^{-\frac{1}{Q}} \gamma_2 \left(\frac{(\mu\otimes\mathcal{L}^1)(A_n)}{\mu(B_n)} \right)^{1+\frac{1}{Q}} \mu(B_n),
\end{aligned} \tag{5.2}$$

where $\gamma_2 = C\gamma_1^{1+\frac{1}{Q}}$. On the other hand, we have

$$\begin{aligned}
\iint_{Q_n^-} ((u(t)-k_n)_-\zeta_n(t))^\kappa \,\mathrm{d}\mu\,\mathrm{d}t &\ge \iint_{Q_{n+1}^-} ((u(t)-k_n)_-\zeta_n(t))^\kappa \,\mathrm{d}\mu\,\mathrm{d}t \\
&\ge \iint_{A_{n+1}} (u(t)-k_n)_-^\kappa \,\mathrm{d}\mu\,\mathrm{d}t \\
&= \iint_{A_{n+1}} (k_n-u(t))^\kappa \,\mathrm{d}\mu\,\mathrm{d}t \\
&\ge \iint_{A_{n+1}} (k_n-k_{n+1})^\kappa \,\mathrm{d}\mu\,\mathrm{d}t \\
&= 2^{-\kappa n} \left(\frac{\omega\xi(1-a)}{2} \right)^\kappa (\mu\otimes\mathcal{L}^1)(A_{n+1}).
\end{aligned} \tag{5.3}$$

Let

$$Y_n = \frac{(\mu\otimes\mathcal{L}^1)(A_n)}{(\mu\otimes\mathcal{L}^1)(Q_n^-)},$$

for $n \in \mathbb{N}$. By (5.3) and (5.2) we obtain

$$\begin{aligned}
Y_{n+1} &= \frac{(\mu\otimes\mathcal{L}^1)(A_{n+1})}{(\mu\otimes\mathcal{L}^1)(Q_{n+1}^-)} \\
&\le \frac{1}{(\mu\otimes\mathcal{L}^1)(Q_{n+1}^-)} 2^{\kappa n} \left(\frac{\omega\xi(1-a)}{2} \right)^{-\kappa} \iint_{Q_n^-} ((u(t)-k_n)_-\zeta_n(t))^\kappa \,\mathrm{d}\mu\,\mathrm{d}t \\
&\le \frac{\mu(B_n)}{(\mu\otimes\mathcal{L}^1)(Q_{n+1}^-)} 2^{\kappa n} \left(\frac{\omega\xi(1-a)}{2} \right)^{-\kappa} \left(2^{n(1+\frac{1}{Q})} \rho^{-\frac{1}{Q}} \gamma_2 \left(\frac{(\mu\otimes\mathcal{L}^1)(A_n)}{\mu(B_n)} \right)^{1+\frac{1}{Q}} \right) \\
&= \frac{\mu(B_n)(\theta_n\rho)^{1+\frac{1}{Q}}}{\mu(B_{n+1})\theta_{n+1}\rho^{1+\frac{1}{Q}}} 2^{n(1+\frac{1}{Q}+\kappa)} \gamma_2 \left(\frac{\omega\xi(1-a)}{2} \right)^{-\kappa} Y_n^{1+\frac{1}{Q}} \\
&\le \frac{\mu(B_n)}{\mu(B_{n+1})} \frac{\theta_n}{\theta_{n+1}} b^n \gamma_3 Y_n^{1+\frac{1}{Q}},
\end{aligned} \tag{5.4}$$

where

$$b = 2^{1+\frac{1}{Q}+\kappa} \quad \text{and} \quad \gamma_3 = \gamma_2 \theta^{\frac{1}{Q}} \left(\frac{\omega\xi(1-a)}{2} \right)^{-\kappa}.$$

By the doubling property we have

$$\mu(B_n) = \mu(B_{\rho_n}(x_0)) \le \mu(B_{2\rho_{n+1}}(x_0)) \le C_\mu \mu(B_{\rho_{n+1}}(x_0)) = \mu(B_{n+1}),$$

and consequently

$$\frac{\mu(B_n)}{\mu(B_{n+1})}\frac{\theta_n}{\theta_{n+1}} \le 2C_\mu,$$

for every $n \in \mathbb{N}$. By (5.4) we conclude

$$Y_{n+1} \le 2C_\mu b^n \gamma_3 Y_n^{1+\frac{1}{Q}} = \gamma_4 b^n Y_n^{1+\frac{1}{Q}},$$

where

$$\gamma_4 = 2C_\mu\gamma_3 = 2^{\frac{Q-1}{Q}} C \left(\frac{\omega\xi}{\theta}\right)^{\frac{1}{Q}} (1-a)^{-\frac{Q+2}{Q}} \left(a\gamma\frac{2(\theta-\overline{\theta})+a\xi\omega}{(\theta-\overline{\theta})}\right)^{\frac{Q+1}{Q}}.$$

By Lemma 5.1, we have $Y_n \to 0$ as $n \to \infty$ provided

$$\begin{aligned} Y_0 &\le \gamma_4^{-Q} b^{-Q^2} = 2^{-(Q-1)} C \left(\frac{\omega\xi}{\theta}\right)^{-1} (1-a)^{Q+2} \left(a\gamma\frac{2(\theta-\overline{\theta})+a\xi\omega}{(\theta-\overline{\theta})}\right)^{-(Q+1)} \\ &= \nu_- = \nu_-(\gamma, C_\mu, C_P, \omega, \xi, a, \theta, \overline{\theta}). \end{aligned}$$

The proof of (ii) is almost identical. One starts from inequalities (4.1) for the truncated functions $(u-k_n)_+$ with $k_n = \mu_+ - \xi_n\omega$ for the same choice of ξ_n. □

6. Time expansion of positivity

In this section we prove an expansion of positivity result, which is a version of [15, Lemma 7.1] on metric measure spaces. Roughly speaking, it asserts that information on the measure of the positivity set of u at time level t_0 over the ball $B_\rho(x_0)$, translates into an expansion of positivity set in time (from t_0 to $t_0+\theta\rho$ for some suitable θ). Most of the arguments and proofs are based on the energy estimates and De Giorgi Lemma of Section 4 and Section 5.

For a cylinder $Q^+_{2\rho,\theta}(x_0,t_0) = B_{2\rho}(x_0) \times (t_0, t_0+\theta\rho) \subset \Omega_T$, let

$$m_+ \ge \operatorname*{ess\,sup}_{Q^+_{2\rho,\theta}(x_0,t_0)} u, \quad m_- \le \operatorname*{ess\,inf}_{Q^+_{2\rho,\theta}(x_0,t_0)} u \quad \text{and} \quad \omega \ge m_+ - m_-.$$

The parameter θ will be determined by the proof. Let $\xi \in (0,1)$ be a fixed parameter.

Lemma 6.1. *Let $u \in DG^-(\Omega_T;\gamma)$ and assume that*

$$\mu(\{x \in B_\rho(x_0) : u(x,t_0) \ge m_- + \xi\omega\}) \ge \frac{1}{2}\mu(B_\rho(x_0)),$$

for some $(x_0,t_0) \in \Omega_T$ and some $\rho > 0$. Then there exist $\delta = \delta(C_\mu,\gamma) \in (0,1)$ and $\varepsilon \in (0,1)$ such that

$$\mu(\{x \in B_\rho(x_0) : u(x,t) \ge m_- + \varepsilon\xi\omega\}) \ge \frac{1}{4}\mu(B_\rho(x_0)),$$

for every $t \in (t_0, t_0+\delta\xi\omega\rho)$.

Proof. Let $A_{k,\rho}(t) = \{x \in B_\rho(x_0) : u(x,t) < k\}$ with $k > 0$ and $t > 0$. Since

$$\begin{aligned} &\frac{1}{2}\mu(B_\rho(x_0)) + \mu(\{x \in B_\rho(x_0) : u(x,t_0) < m_- + \xi\omega\}) \\ &\quad \le \mu(\{x \in B_\rho(x_0) : u(x,t_0) \ge m_- + \xi\omega\}) + \mu(\{x \in B_\rho(x_0) : u(x,t_0) < m_- + \xi\omega\}) \\ &\quad \le \mu(B_\rho(x_0)), \end{aligned}$$

we have

$$\mu(A_{m_-+\xi\omega,\rho}(t_0)) = \mu(\{x \in B_\rho(x_0) : u(x,t_0) < m_- + \xi\omega\}) \le \frac{1}{2}\mu(B_\rho(x_0)).$$

Let ζ be a Lipschitz cutoff function which is independent of t, $0 \le \zeta \le 1$, $\zeta = 1$ in $B_{(1-\sigma)\rho}(x_0)$ and $g_\zeta \le \frac{1}{\sigma\rho}$, where $\sigma \in (0,1)$ is to be chosen. We apply the De Giorgi condition for $(u-(m_-+\xi\omega))_-$ in $Q^+_{\rho,\theta}(x_0,t_0) = B_\rho(x_0) \times (t_0, t_0+\theta\rho]$, where $\theta > 0$ is to be chosen, and obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_0 \le t \le t_0+\theta\rho} \int_{B_\rho(x_0)} \zeta(t)(u(t)-(m_-+\xi\omega))_-^2 \,\mathrm{d}\mu + \int_{t_0}^{t_0+\theta\rho} \|D((u-(m_-+\xi\omega))_-\zeta)(t)\|(B_\rho(x_0))\,\mathrm{d}t \\
&\quad\le \gamma \iint_{Q^+_{\rho,\theta}(x_0,t_0)} g_\zeta(t)(u(t)-(m_-+\xi\omega))_- \,\mathrm{d}\mu\,\mathrm{d}t - \left[\int_{B_\rho(x_0)} (u(t)-(m_-+\xi\omega))_-^2\zeta(t)\,\mathrm{d}\mu\right]_{t=t_0}^{t_0+\theta\rho} \\
&\quad\le \gamma \iint_{Q^+_{\rho,\theta}(x_0,t_0)} g_\zeta(t)(u(t)-(m_-+\xi\omega))_- \,\mathrm{d}\mu\,\mathrm{d}t + \int_{B_\rho(x_0)} (u(t_0)-(m_-+\xi\omega))_-^2\zeta(t_0)\,\mathrm{d}\mu \\
&\quad\le \frac{\gamma}{\sigma\rho} \iint_{Q^+_{\rho,\theta}(x_0,t_0)} (u(t)-(m_-+\xi\omega))_- \,\mathrm{d}\mu\,\mathrm{d}t + \int_{B_\rho(x_0)} (u(t_0)-(m_-+\xi\omega))_-^2\,\mathrm{d}\mu.
\end{aligned}$$

Notice that for $t \in (t_0, t_0+\theta\rho)$, we have

$$\begin{aligned}
&\int_{B_{(1-\sigma)\rho}(x_0)} (u(t)-(m_-+\xi\omega))_-^2\zeta(t)\,\mathrm{d}\mu = \int_{B_{(1-\sigma)\rho}(x_0)} (u(t)-(m_-+\xi\omega))_-^2\,\mathrm{d}\mu \\
&\quad\le \int_{B_\rho(x_0)} (u(t)-(m_-+\xi\omega))_-^2\zeta(t)\,\mathrm{d}\mu \\
&\quad\le \operatorname*{ess\,sup}_{t_0 \le t \le t_0+\theta\rho} \int_{B_\rho(x_0)} \zeta(t)(u(t)-(m_-+\xi\omega))_-^2\,\mathrm{d}\mu \\
&\quad\le \frac{\gamma}{\sigma\rho} \iint_{Q^+_{\rho,\theta}(x_0,t_0)} (u(t)-(m_-+\xi\omega))_- \,\mathrm{d}\mu\,\mathrm{d}t + \int_{B_\rho(x_0)} (u(t_0)-(m_-+\xi\omega))_-^2\,\mathrm{d}\mu \\
&\quad\le \frac{\gamma}{\sigma\rho}\xi\omega\theta\rho\mu(B_\rho(x_0)) + \frac{(\xi\omega)^2}{2}\mu(B_\rho(x_0)) \\
&\quad= (\xi\omega)^2\left(\frac{\gamma\theta}{\sigma(\xi\omega)} + \frac{1}{2}\right)\mu(B_\rho(x_0)).
\end{aligned}$$

The last inequality holds, because

$$\begin{aligned}
\int_{B_\rho(x_0)} (u(t_0)-(m_-+\xi\omega))_-^2\,\mathrm{d}\mu &= \int_{A_{m_-+\xi\omega,\rho}(t_0)} (m_-+\xi\omega-u(t_0))^2\,\mathrm{d}\mu \\
&\le (\xi\omega)^2\mu(A_{m_-+\xi\omega,\rho}(t_0)) \le \frac{(\xi\omega)^2}{2}\mu(B_\rho(x_0)),
\end{aligned}$$

and

$$\begin{aligned}
\iint_{Q^+_{\rho,\theta}(x_0,t_0)} (u(t)-(m_-+\xi\omega))_-\,\mathrm{d}\mu\,\mathrm{d}t &= \int_{t_0}^{t_0+\theta\rho}\int_{A_{m_-+\xi\omega,\rho}(t_0)} (m_-+\xi\omega-u(t))\,\mathrm{d}\mu\,\mathrm{d}t \\
&\le \int_{t_0}^{t_0+\theta\rho} \xi\omega \int_{A_{m_-+\xi\omega,\rho}(t_0)} \mathrm{d}\mu\,\mathrm{d}t \\
&\le \xi\omega\mu(B_\rho(x_0))\theta\rho.
\end{aligned}$$

Therefore

$$\int_{B_{(1-\sigma)\rho}(x_0)} (u(t)-(m_-+\xi\omega))_-^2\,\mathrm{d}\mu \le (\xi\omega)^2\left(\frac{\gamma\theta}{\sigma(\xi\omega)} + \frac{1}{2}\right)\mu(B_\rho(x_0)),$$

for every $t \in (t_0, t_0 + \theta\rho)$. The left-hand side can be estimated by

$$\begin{aligned}\int_{B_{(1-\sigma)\rho}(x_0)} (u(t) - (m_- + \xi\omega))_-^2 \,\mathrm{d}\mu &\geq \int_{A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)} (u(t) - (m_- + \xi\omega))_-^2 \,\mathrm{d}\mu \\ &\geq \int_{A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)} (m_- + \xi\omega - u(t))^2 \,\mathrm{d}\mu \\ &> \int_{A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)} (m_- + \xi\omega - (m_- + \varepsilon\xi\omega))^2 \,\mathrm{d}\mu \\ &= (\xi\omega)^2(1-\varepsilon)^2 \mu(A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)),\end{aligned}$$

where $\varepsilon \in (0, 1)$ is to be chosen.

By the doubling property and Bernoulli's inequality, we obtain

$$\begin{aligned}\mu(A_{m_-+\varepsilon\xi\omega,\rho}(t)) &= \mu\left(A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t) \cup \left(A_{m_-+\varepsilon\xi\omega,\rho}(t) \setminus A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)\right)\right) \\ &\leq \mu\left(A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)\right) + \mu(B_\rho(x_0) \setminus B_{(1-\sigma)\rho}(x_0)) \\ &\leq \mu\left(A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)\right) + \left(\mu(B_\rho(x_0)) - \frac{\mu(B_{(1-\sigma)\rho}(x_0))}{\mu(B_\rho(x_0))}\mu(B_\rho(x_0))\right) \\ &\leq \mu\left(A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)\right) + \mu(B_\rho(x_0))\left(1 - C_\mu^2(1-\sigma)^Q\right) \\ &\leq \mu\left(A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)\right) + \mu(B_\rho(x_0))\left(1 - (1-\sigma)^Q\right) \\ &\leq \mu\left(A_{m_-+\varepsilon\xi\omega,(1-\sigma)\rho}(t)\right) + Q\sigma\mu(B_\rho(x_0)).\end{aligned}$$

Combining these estimates gives

$$\begin{aligned}\mu(A_{m_-+\varepsilon\xi\omega,\rho}(t)) &\leq \frac{1}{(\xi\omega)^2(1-\varepsilon)^2} \int_{B_{(1-\sigma)\rho}(x_0)} (u(t) - (m_- + \xi\omega))_-^2 \,\mathrm{d}\mu + Q\sigma\mu(B_\rho(x_0)) \\ &\leq \frac{1}{(\xi\omega)^2(1-\varepsilon)^2}\left((\xi\omega)^2\left(\frac{\gamma\theta}{\sigma\xi\omega} + \frac{1}{2}\right)\mu(B_\rho(x_0))\right) + Q\sigma\mu(B_\rho(x_0)) \\ &= \mu(B_\rho(x_0))\left(\frac{1}{(1-\varepsilon)^2}\left(\frac{\gamma\theta}{\sigma\xi\omega} + \frac{1}{2}\right) + Q\sigma\right) \\ &\leq \frac{\mu(B_\rho(x_0))}{(1-\varepsilon)^2}\left(\frac{\gamma\theta}{\sigma\xi\omega} + \frac{1}{2} + Q\sigma\right).\end{aligned}$$

Setting $\theta = \frac{\xi\omega}{2^8\gamma Q}$ and $\sigma = \frac{1}{16Q}$, we obtain

$$\frac{1}{(1-\varepsilon)^2}\left(\frac{\gamma\theta}{\sigma\xi\omega} + \frac{1}{2} + Q\sigma\right) = \frac{1}{(1-\varepsilon)^2}\left(\frac{\frac{1}{2^8Q}}{\frac{1}{16Q}} + \frac{1}{2} + \frac{1}{16}\right) = \frac{1}{(1-\varepsilon)^2}\frac{5}{8} < \frac{3}{4}.$$

By letting $0 < \varepsilon \leq \frac{1}{32}$, we have

$$\mu(\{x \in B_\rho(x_0) : u(x,t) > m_- + \varepsilon\xi\omega\}) + \mu(A_{m_-+\varepsilon\xi\omega,\rho}(t)) \geq \mu(B_\rho(x_0)),$$

and thus

$$\begin{aligned}\mu(\{x \in B_\rho(x_0) : u(x,t) > m_- + \varepsilon\xi\omega\}) &\geq \mu(B_\rho(x_0)) - \mu(A_{m_-+\varepsilon\xi\omega,\rho}(t)) \\ &\geq \mu(B_\rho(x_0)) - \frac{3}{4}\mu(B_\rho(x_0)) = \frac{1}{4}\mu(B_\rho(x_0)).\end{aligned}$$

Therefore, the claim holds for $\delta = \frac{1}{2^8\gamma Q}$. $\square$

7. Characterization of continuity

Finally we are ready to prove the main result of this paper.

Theorem 7.1. *Let $u \in L^1_{\mathrm{loc}}(0,T;BV_{\mathrm{loc}}(\Omega))$ be a variational solution to the total variation flow in Ω_T. Then u is continuous at some $(x_0,t_0) \in \Omega_T$ if and only if*

$$\lim_{\rho\to 0+} \frac{\rho}{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,1}(x_0,t_0))}\int_{t_0-\rho}^{t_0} \|Du(t)\|(B_\rho(x_0))\,\mathrm{d}t = 0.$$

Proof. We begin with the necessary part of Theorem 7.1. By Proposition 4.2, we have $u \in DG(\Omega_T;\gamma)$ with $\gamma = 8$. Assume that u is continuous at $(x_0,t_0)\in\Omega_T$. Without loss of generality we may assume $u(x_0,t_0)=0$. Let ζ be a Lipschitz cutoff function with $0\le\zeta\le 1$, $\zeta = 0$ on $(X\times\mathbb{R})\setminus Q^-_{2\rho,1}(x_0,t_0)$, $\zeta = 1$ on $Q^-_{\frac{3}{2}\rho,1}(x_0,t_0)$, $\zeta(\cdot,t_0-2\rho)=0$, $\zeta_t\ge 0$ and $g_\zeta+\zeta_t\le\frac{3}{\rho}$. We apply (4.1) with $\theta=1$, $k=0$ and neglect the supremum term of the left-hand side to obtain

$$\int_{t_0-2\rho}^{t_0}\|D(u_+\zeta)(t)\|(B_{2\rho}(x_0))\,\mathrm{d}t \le \gamma\iint_{Q^-_{2\rho,1}(x_0,t_0)}(u_+(t)g_\zeta(t)+u_+(t)^2|\zeta_t(t)|)\,\mathrm{d}\mu\,\mathrm{d}t,$$

and

$$\int_{t_0-2\rho}^{t_0}\|D(u_-\zeta)(t)\|(B_{2\rho}(x_0))\,\mathrm{d}t \le \gamma\iint_{Q^-_{2\rho,1}(x_0,t_0)}(u_-(t)g_\zeta(t)+u_-(t)^2|\zeta_t(t)|)\,\mathrm{d}\mu\,\mathrm{d}t.$$

By adding up the inequalities above and using the doubling property of the measure, we obtain

$$\begin{aligned}\int_{t_0-2\rho}^{t_0}\|D(u\zeta)(t)\|(B_{2\rho}(x_0))\,\mathrm{d}t &\le \gamma\iint_{Q^-_{2\rho,1}(x_0,t_0)}(g_\zeta(t)|u(t)|+|\zeta_t(t)|u(t)^2)\,\mathrm{d}\mu\,\mathrm{d}t\\ &\le \frac{3\gamma}{\rho}\iint_{Q^-_{2\rho,1}(x_0,t_0)}(|u(t)|+u(t)^2)\,\mathrm{d}\mu\,\mathrm{d}t\\ &= \frac{3\gamma}{\rho}(\mu\otimes\mathcal{L}^1)(Q^-_{2\rho,1}(x_0,t_0))⨍⨍_{Q^-_{2\rho,1}(x_0,t_0)}(|u(t)|+u(t)^2)\,\mathrm{d}\mu\,\mathrm{d}t\\ &\le \frac{6C_\mu\gamma}{\rho}(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,1}(x_0,t_0))⨍⨍_{Q^-_{2\rho,1}(x_0,t_0)}(|u(t)|+u(t)^2)\,\mathrm{d}\mu\,\mathrm{d}t.\end{aligned}$$

Since $u\zeta = u$ in $Q^-_{\frac{3}{2}\rho,1}(x_0,t_0)\supseteq Q^-_{\rho,1}(x_0,t_0)$, we obtain

$$\begin{aligned}&\frac{\rho}{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,1}(x_0,t_0))}\int_{t_0-2\rho}^{t_0}\|Du(t)\|(B_\rho(x_0))\,\mathrm{d}t\\ &\quad\le \frac{\rho}{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,1}(x_0,t_0))}\int_{t_0-2\rho}^{t_0}\|D(u\zeta)(t)\|(B_{2\rho}(x_0))\,\mathrm{d}t\\ &\quad\le 6C_\mu\gamma⨍⨍_{Q^-_{2\rho,1}(x_0,t_0)}(|u(t)|+u(t)^2)\,\mathrm{d}\mu\,\mathrm{d}t.\end{aligned}$$

The right-hand side tends to zero as $\rho\to 0$ implying the necessary condition of Theorem 7.1.

Let us then prove the sufficient part of Theorem 7.1. Let $(x_0,t_0)\in\Omega_T$ and let $\rho>0$ be so small that $Q^-_{\rho,1}(x_0,t_0) = B_\rho(x_0)\times(t_0-\rho,t_0]\subset\Omega_T$. Set

$$m_+ = \operatorname*{ess\,sup}_{Q^-_{\rho,1}(x_0,t_0)} u,\quad m_- = \operatorname*{ess\,inf}_{Q^-_{\rho,1}(x_0,t_0)} u\quad\text{and}\quad \omega = m_+-m_- = \operatorname*{ess\,osc}_{Q^-_{\rho,1}(x_0,t_0)} u.$$

Without loss of generality we may assume that $\omega \le 1$ so that

$$Q^-_{\rho,\omega}(x_0,t_0) = B_\rho(x_0) \times (t_0 - \omega\rho, t_0] \subset Q^-_{\rho,1}(x_0,t_0) \subset \Omega_T.$$

Therefore,

$$\operatorname*{ess\,inf}_{Q^-_{\rho,\omega}(x_0,t_0)} u \ge m_-, \qquad \operatorname*{ess\,sup}_{Q^-_{\rho,\omega}(x_0,t_0)} u \le m_+ \quad \text{and} \quad \omega \ge \operatorname*{ess\,osc}_{Q^-_{\rho,\omega}(x_0,t_0)} u.$$

For a contradiction, assume that u is not continuous at (x_0,t_0). Then there exists $\rho_0 > 0$ and $\omega_0 > 0$ such that

$$\omega_{\tilde\rho} = \operatorname*{ess\,osc}_{Q^-_{\tilde\rho,1}(x_0,t_0)} u \ge \omega_0 > 0,$$

for all $0 < \tilde\rho \le \rho_0$. Let $\tilde\delta = \frac{1}{2^8\gamma Q}$, determined as in the proof of Lemma 6.1 at the time level $\tilde t = t_0 - \frac{\tilde\delta\omega\rho}{2}$. Clearly

$$\mu\left(\left\{x \in B_\rho(x_0) : u\big(x, t_0 - \tfrac{\tilde\delta\omega\rho}{2}\big) \ge m_- + \frac{\omega}{2}\right\}\right) \ge \frac{1}{2}\mu(B_\rho(x_0)),$$

or

$$\mu\left(\left\{x \in B_\rho(x_0) : u\big(x, t_0 - \tfrac{\tilde\delta\omega\rho}{2}\big) \le m_+ + \frac{\omega}{2}\right\}\right) \ge \frac{1}{2}\mu(B_\rho(x_0)).$$

Assuming the former holds, by Lemma 6.1 there is a δ, actually $\delta = \tilde\delta$ works, and $\varepsilon = \frac{1}{32}$ such that

$$\mu\left(\left\{x \in B_\rho(x_0) : u(x,t) \ge m_- + \frac{\omega}{64}\right\}\right) \ge \frac{1}{4}\mu(B_\rho(x_0)),$$

for every $t \in (t_0 - \frac{\tilde\delta\omega\rho}{2}, t_0]$. Let $2\tilde\xi = \frac{1}{64}\tilde\delta$. Since $\frac{1}{64}\omega \ge \frac{1}{64}\tilde\delta\omega$, then

$$\left\{x \in B_\rho(x_0) : u(x,t) \ge m_- + \frac{\omega}{64}\right\} \subset \left\{x \in B_\rho(x_0) : u(x,t) \ge m_- + \frac{\tilde\delta\omega}{64}\right\},$$

and thus

$$\mu(\{x \in B_\rho(x_0) : u(x,t) > m_- + 2\tilde\xi\omega\}) \ge \mu\left(\left\{x \in B_\rho(x_0) : u(x,t) \ge m_- + \frac{\omega}{64}\right\}\right) \ge \frac{1}{4}\mu(B_\rho(x_0)),$$

for every $t \in (t_0 - \frac{\tilde\delta\omega\rho}{2}, t_0]$. Since $(t_0 - \tilde\xi\omega\rho, t_0] \subset (t_0 - \frac{\tilde\delta\omega\rho}{2}, t_0]$, we have

$$\mu(\{x \in B_\rho(x_0) : u(x,t) > m_- + 2\tilde\xi\omega\}) \ge \frac{1}{4}\mu(B_\rho(x_0)), \tag{7.1}$$

for every $t \in (t_0 - \tilde\xi\omega\rho, t_0]$. Next, we apply Lemma 2.6 to the function $u(\cdot,t)$, for t in the range $(t_0 - \tilde\xi\omega\rho, t_0]$ over the ball $B_\rho(x_0)$ with $k = m_- + \tilde\xi\omega$ and $l = m_- + 2\tilde\xi\omega$, so that $l - k = \tilde\xi\omega$.

By the doubling property of the measure and (7.1), we have

$$\begin{aligned}\frac{\tilde\xi\omega}{2^{Q+2}C_\mu} \le \frac{\tilde\xi\omega}{4}\frac{\mu(B_\rho(x_0))}{\mu(B_{2\rho}(x_0))} &\le \frac{\tilde\xi\omega\mu(\{x \in B_\rho(x_0) : u(x,t) > m_- + 2\tilde\xi\omega\})}{\mu(B_{2\rho}(x_0))} \\ &\le C\rho\frac{\|Du(t)\|(\{x \in B_\rho(x_0) : u(x,t) > m_- + \tilde\xi\omega\})}{\mu(\{x \in B_\rho(x_0) : u(x,t) < m_- + \tilde\xi\omega\})}.\end{aligned}$$

This implies

$$\tilde\xi\omega\mu(\{x \in B_\rho(x_0) : u(x,t) < m_- + \tilde\xi\omega\}) \le C\rho\|Du(t)\|(\{x \in B_\rho(x_0) : u(x,t) > m_- + \tilde\xi\omega\}),$$

where $C = C(C_\mu, C_P)$.

Integrating over the time interval $(t_0 - \tilde{\xi}\omega\rho, t_0]$ gives

$$\begin{aligned}
&\tilde{\xi}\omega \int_{t_0-\tilde{\xi}\omega\rho}^{t_0} \mu(\{x \in B_\rho(x_0) : u(x,t) < m_- + \tilde{\xi}\omega\})\,\mathrm{d}t \\
&\qquad \le C\rho \int_{t_0-\tilde{\xi}\omega\rho}^{t_0} \|Du(t)\|(\{x \in B_\rho(x_0) : u(x,t) > m_- + \tilde{\xi}\omega\})\,\mathrm{d}t \\
&\qquad \le C\rho \int_{t_0-\tilde{\xi}\omega\rho}^{t_0} \|Du(t)\|(B_\rho(x_0))\,\mathrm{d}t \\
&\qquad = C\rho \frac{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0))}{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0))} \int_{t_0-\tilde{\xi}\omega\rho}^{t_0} \|Du(t)\|(B_\rho(x_0))\,\mathrm{d}t \\
&\qquad = \frac{C\rho}{\tilde{\xi}\omega} \frac{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0))}{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,1}(x_0,t_0))} \int_{t_0-\tilde{\xi}\omega\rho}^{t_0} \|Du(t)\|(B_\rho(x_0))\,\mathrm{d}t.
\end{aligned}$$

Since

$$\int_{t_0-\tilde{\xi}\omega\rho}^{t_0} \mu(\{x \in B_\rho(x_0) : u(x,t) < m_- + \tilde{\xi}\omega\})\,\mathrm{d}t \ge (\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0) \cap \{u < m_- + \tilde{\xi}\omega\}),$$

we have

$$\begin{aligned}
&\frac{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0) \cap \{u < m_- + \tilde{\xi}\omega\})}{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0))} \\
&\qquad \le \frac{C}{(\tilde{\xi}\omega_0)^2} \frac{\rho}{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,1}(x_0,t_0))} \int_{t_0-\tilde{\xi}\omega\rho}^{t_0} \|Du(\cdot,t)\|(B_\rho(x_0))\,\mathrm{d}t.
\end{aligned}$$

By assumption, the right-hand side tends to zero as $\rho \to 0+$. Hence, there exists $\rho > 0$ small enough such that

$$\frac{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0) \cap \{u < m_- + \tilde{\xi}\omega\})}{(\mu\otimes\mathcal{L}^1)(Q^-_{\rho,\tilde{\xi}\omega}(x_0,t_0))} \le \nu_-,$$

where ν_- is the number in Lemma 5.2 for such a choice of parameters. Lemma 5.2 implies $u \ge m_- + \frac{1}{2}\tilde{\xi}\omega$ $(\mu\otimes\mathcal{L}^1)$-almost everywhere in $Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)$ and consequently

$$\operatorname*{ess\,inf}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u \ge m_- + \frac{\tilde{\xi}\omega}{2}.$$

This implies

$$\begin{aligned}
\operatorname*{ess\,osc}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u &= \operatorname*{ess\,sup}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u - \operatorname*{ess\,inf}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u \le \operatorname*{ess\,sup}_{Q^-_{\rho,1}(x_0,t_0)} u - m_- - \frac{\tilde{\xi}\omega}{2} \\
&= m_+ - m_- - \frac{\tilde{\xi}\omega}{2} = \omega - \frac{\tilde{\xi}\omega}{2} = \left(1 - \frac{\tilde{\xi}}{2}\right)\omega = \eta\omega,
\end{aligned}$$

for some $\eta \in (0,1)$. With $\rho_1 = \frac{1}{2}\tilde{\xi}\omega\rho$ we have $Q^-_{\rho_1,1}(x_0,t_0) \subset Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)$ and thus

$$\operatorname*{ess\,sup}_{Q^-_{\rho_1,1}(x_0,t_0)} u \le \operatorname*{ess\,sup}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u \quad \text{and} \quad \operatorname*{ess\,inf}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u \le \operatorname*{ess\,sup}_{Q^-_{\rho_1,1}(x_0,t_0)} u.$$

Therefore, we have

$$\begin{aligned}\omega_{\rho_1} &= \operatorname*{ess\,osc}_{Q^-_{\rho_1,1}(x_0,t_0)} u = \operatorname*{ess\,sup}_{Q^-_{\rho_1,1}(x_0,t_0)} u - \operatorname*{ess\,inf}_{Q^-_{\rho_1,1}(x_0,t_0)} u \\ &\le \operatorname*{ess\,sup}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u - \operatorname*{ess\,inf}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} u = \operatorname*{ess\,osc}_{Q^-_{\frac{1}{2}\rho,\tilde{\xi}\omega}(x_0,t_0)} \le \eta\omega.\end{aligned}$$

By repeating the same argument starting from the cylinder $Q^-_{\rho_1,1}(x_0,t_0)$ and proceeding recursively, we generate a decreasing sequence of radii $\rho_n \to 0$ such that

$$\omega_0 \le \operatorname*{ess\,osc}_{Q^-_{\rho_n,1}(x_0,t_0)} u \le \eta^n\omega,$$

for every $n \in \mathbb{N}$. This is a contradiction with the assumption u is not continuous at (x_0,t_0). □

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