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**ORIGINAL PAPER** 





# **Rational functions as new variables**

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## Abstract

In *multicentric calculus*, one takes a polynomial p with distinct roots as a new variable and represents complex valued functions by  $\mathbb{C}^d$ -valued functions, where d is the degree of p. An application is e.g. the possibility to represent a piecewise constant holomorphic function as a convergent power series, simultaneously in all components of  $|p(z)| \leq \rho$ . In this paper, we study the necessary modifications needed, if we take a rational function r = p/q as the new variable instead. This allows to consider functions defined in neighborhoods of any compact set as opposed to the polynomial case where the domains  $|p(z)| \leq \rho$  are always polynomially convex. Two applications are formulated. One giving a convergent power series expression for Sylvester equations AX - XB = C in the general case of A, B being bounded operators in Banach spaces with distinct spectra. The other application formulates a K-spectral result for bounded operators in Hilbert spaces.

Keywords Rational functions · Series expansions · Functional calculus

Mathematics Subject Classification  $~30B10\cdot 30C10\cdot 30E99\cdot 46J10\cdot 47A25\cdot 47A60$ 

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## 1 Introduction

In a series of papers [7–10] one of us has considered the possibility and applications of taking a polynomial with simple zeros as a new *global* variable w = p(z). As the polynomial of degree *d* is not one-to-one, complex valued scalar functions  $\varphi$  are represented by  $\mathbb{C}^d$ -valued functions *f*. Additionally [4] contains modifications to the case  $w = p(z)^n$  and [1] discusses extensions to n-tuples of operators.

The key idea in applications to functional calculus is to have a polynomial p such that p(A) is either small so that the series expansions of f converge fast at p(A), or "structurally simpler" than A so that, for example, a matrix A with non-trivial Jordan blocks becomes diagonalizable.

By Hilbert's lemniscate theorem, see e.g. [11], any *polynomially convex* compact set can be approximated from outside arbitrarily well using polynomial lemniscates  $|p(z)| = \rho$ . Taking such a polynomial as a new variable maps the analysis from inside the lemniscate into a disc, where a lot of analysis tools are available. At the end one transforms the results back into scalar functions in the original variable.

Sometimes one needs to have a representation for a function in sets which are not polynomially convex. To that end it is natural to ask whether taking a rational function in place of the polynomial leads to a useful representation in such cases. It turns out that choosing a rational function r = p/q with q of lower degree than p much of the multicentric calculus carries over with minor modifications.

The paper is organized as follows. We shall first formulate and prove a "rational lemniscate lemma" approximating any compact set arbitrarily well *in a fixed neighborhood of it.* This is done in Sect. 2. We also formulate a result as corollary where the spectra of bounded operators play the role of the compact set.

In Sect. 3 we consider the existence and uniqueness of the representations using rational functions as variables. Given  $\varphi$  there exists a unique representing function f excluding critical points of the rational function and if  $\varphi$  is holomorphic then the singularities of f at critical values are removable, so f is holomorphic as well. In order to determine the largest class of functions for which the representation is continuous at critical values we modify the approach in [9] by moving the focus into the functions f and construct a unital Banach algebra for such functions so that the original function  $\varphi$  appears as the Gelfand transform of f, bringing the wealth of Banach algebra theory available. Throughout the paper we use the convention that by function holomorphic in a compact set we mean that it is holomorphic in some unknown neighborhood of it.

We shall indicate two applications in which we remove the assumption on the compact set to be polynomially convex, needed when using polynomials as new variables. Sylvester equation AX - XB = C with bounded operators in Banach spaces, has a unique solution for every *C* if and only if the spectra are separated:  $\sigma(A) \cap \sigma(B) = \emptyset$ . We show that then, without any other assumptions, there exists a rational function such that the solution to the Sylvester equation can be represented as a convergent power series. This is discussed in Sect. 4. This generalizes a result of [10] where a similar statement was shown for polynomials with the extra assumption that the polynomial convex hulls of the spectra do not intersect.

In [8] it was shown that polynomial lemniscates provide *K*-spectral sets and we generalize the discussion in Sect. 5 for rational lemniscates.

# 2 Rational lemniscate sets

### 2.1 Approximating compact sets with rational lemniscates

Hilbert Lemniscate Theorem, e.g. [11], provides the existence of a polynomial such that it can surround any polynomially compact set arbitrarily closely. In fact, given a compact K such that  $\mathbb{C} \setminus K$  is *simply connected* and  $\varepsilon > 0$  there exists a polynomial p such that if

$$V_p = \{ z \in \mathbb{C} : |p(z)| \le 1 \}$$

and  $K_{\varepsilon} = \{z : \operatorname{dist}(z, K) \leq \varepsilon\}$ , then

 $K \Subset V_p \Subset K_{\varepsilon}.$ 

Here  $\in$  means that the smaller compact is included in the interior of the larger compact.

Suppose r = p/q is a rational function, with p and q having no common roots. Again we put

$$V_r = \{ z \in \mathbb{C} : |r(z)| \le 1 \}$$

$$\tag{1}$$

but we need to restrict  $V_r$  into a compact set as we do not have control of the size of r globally. To that end we denote by  $\Gamma_{\epsilon}$  the following compact set surrounding K:

$$\Gamma_{\varepsilon} := K_{\varepsilon} \setminus \operatorname{int} K_{\varepsilon/2}.$$
<sup>(2)</sup>

**Theorem 2.1** Given a compact  $K \subset \mathbb{C}$  and  $\varepsilon > 0$  let  $\Gamma_{\varepsilon}$  be as in (2). Then there exists a rational function r such that

$$K \Subset V_r$$
 and  $V_r \cap \Gamma_{\varepsilon} = \emptyset$ . (3)

*Further, the rational function* r = p/q *can be so chosen that* deg  $q < \deg p$ .

**Proof** We define a piecewise constant holomorphic function  $\chi$  such that it vanishes in some small neighborhood of *K* and equals 2 in a small neighborhood of  $\Gamma_{\varepsilon}$ . Denoting  $E = K \cup \Gamma_{\varepsilon}$  we can approximate  $\chi$  by Runge's Theorem, [5], with rational functions in *E* uniformly. In particular there exists a rational function  $r_0$  such that

$$\max_{z \in E} |\chi(z) - r_0(z)| < 1/2.$$

Then in *K* we have  $|r_0(z)| < 1/2$  while in  $\Gamma_{\varepsilon}$  we have  $|r_0(z)| > 3/2$ . Thus (3) holds.

In order to show that we can have  $r \to \infty$  as  $z \to \infty$ , denote  $P(z) = 1 - (z/R)^n$ . If  $r_0$  is a rational function satisfying (3) then with *R* and *n* large enough the rational function  $r = Pr_0$  still satisfies (3) with deg  $q < \deg p$ .

As  $V_r$  consists of a finite number of components, bounded by the degree of p there is a finite number of components, each intersecting with K and "surrounded" by  $\Gamma_{\varepsilon}$ . Additionally  $V_r$  may have components both "inside and outside" of  $\Gamma_{\varepsilon}$ . Let us denote by  $V_r(K)$  the union of the components of  $V_r$  which intersect with K, so that in particular  $K \in V_r(K)$ . Assume now that  $\varphi$  is a holomorphic function in some neighborhood of K. Then with small enough  $\varepsilon$  there exists r such that  $\varphi$  is holomorphic in  $V_r(K)$ . Denote by  $\gamma$  the boundary of  $V_r(K)$ , consisting of a finite number of piecewise smooth loops, and oriented so that K stays on the left. Then by Cauchy's theorem we have for  $z \in K$ 

$$\varphi(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi(\lambda)}{\lambda - z} \, \mathrm{d}\lambda. \tag{4}$$

Observe that along  $\gamma$  we have |r(z)| = 1 and thus  $V_r(K)$  is mapped in w = r(z) onto the unit disc - and the scalar function  $\varphi: V_r(K) \to \mathbb{C}$  is likewise replaced by a vectorvalued holomorphic function  $f: \overline{\mathbb{D}} \to \mathbb{C}^d$ . In order to achieve this, we shall decompose the Cauchy kernel into pieces, each yielding one component  $f_i$  of f. Notice that  $r^{-1}(\overline{\mathbb{D}}) = V_r$  may contain components which do not intersect K. However, we have the possibility to define  $\varphi = 0$  in those components and thus the integration and analysis could be done in the whole  $V_r$  as well, if so wanted.

#### 2.2 Spectrum as the compact set

Assume given a bounded operator *A* in a Banach space  $\mathcal{X}, A \in B(\mathcal{X})$ . Fix  $\varepsilon > 0$  and let r = p/q be as in Theorem 2.1 when the spectrum  $\sigma(A)$  is taken as the compact set *K*. In particular *r* is holomorphic in the spectrum and r(A) is a well defined bounded operator. Then

$$||r(A)^{m}||^{1/m} \to \rho(r(A)) = \sup_{z \in \sigma(A)} |r(z)|.$$

Since *r* is not a constant, and  $\sigma(A) \in V_r$ , we have by maximum principle

$$\sup_{z \in \sigma(A)} |r(z)| < 1$$

But then there exists *n* such that  $||r(A)^n|| < 1$ . Denote by  $\tilde{p}$  a tiny perturbation of  $p^n$  so that all roots of  $\tilde{r} = \tilde{p}/q^n$  are simple and we still have  $||\tilde{r}(A)|| < 1$ . In order to formulate the corollary, let us denote by  $\Gamma_{\varepsilon}$  the set surrounding the spectrum as in (2) with  $\sigma(A) = K$ .

**Corollary 2.2** Given a bounded operator  $A \in B(\mathcal{X})$ , fix an  $\varepsilon > 0$  and denote by  $\Gamma_{\varepsilon}$  the set around the spectrum  $\sigma(A)$  as above. Then there exists a rational function



Fig. 1 A rational function of degree 16 separting two circles

r = p/q, such that deg  $q < \deg p$  where p has simple roots and ||r(A)|| < 1, while |r(z)| > 1 for  $z \in \Gamma_{\epsilon}$ .

A typical application of using polynomials or rational functions as new variables is the possibility to deal with piecewise constant holomorphic functions. We mention two natural situations.

**Example 2.3** If the lemniscate set covering the spectrum has several components, then defining the holomorphic function to be identically 1 in one component while setting it 0 in the others leads to an explicit power series representation for the Riesz spectral projection [4]. In Fig. 1 we have a model situation which cannot be obtained by polynomial lemniscates. Two circles are separated from each others with a rational function r = p/q with p of degree 16 and q of degree 9. The set in which

|r(z)| < 1 is white in the picture and dots denote the zeros of *p* while small circles denote the zeros of *q*.

**Example 2.4** Another natural piecewise holomorphic function is the sign-function which equals 1 in the right half plane and -1 in the left. In Sect. 4 we outline a use of it in the solving of the Sylvester equation. Here one is after a polynomial or rational function such that the lemniscate has components both on the left and right half planes without intersecting the imaginary axis. In [4] sets consisting of two intervals, parallel to the imaginary axis and symmetrically located around the origin, were considered as test sets to be separated. As the phenomenon is scaling invariant, the angle  $\alpha$  was used to parametrize the sets

$$L_{\alpha} = \{x + iy : x \in \{-1, 1\}, |y| \le \tan(\alpha)\}.$$

Polynomials were then searched such that  $L_{\alpha} \subset V_p$  while  $V_p \cap i\mathbb{R} = \emptyset$ . With  $p(z) = z^2 - 1$ , suitably scaled, any angle below 45° is clearly possible. With degree 4 one finds polynomials with angle above 61° but the required degree seemed to grow quite fast with  $\alpha$ . For example, angles above 72° were found only with polynomials of degree 14 or higher. For details, see [4]. As to be expected, with rational functions the separation is easier and for example with d = 2 and r(z) = z - 1/z the largest angle, see Fig. 3 below, is already about 69°. In order to have a simple rational function of degree 4 consider

$$r(z) = \frac{z^4 - 2z^2 + 9}{z^3 + 3z} \tag{5}$$

which vanishes at  $\pm \sqrt{2} \pm i$  and has poles at the origin and at  $\pm i\sqrt{3}$ . In Fig. 2 the lemniscate is drawn at the level |r(z)| = 5.6 with  $\alpha = 80^{\circ}$ . A careful numerical search shows that the angle stays below 81° for all rational functions with *p* of degree 4 and *q* of degree 3.

#### 3 Representation using rational functions as variables

Let r = p/q with *p* having *d* simple roots  $\lambda = {\lambda_j}$  so that  $q(\lambda_j) \neq 0$  and such that  $d = \deg p > \deg q$ . Denoting by  $\delta_i$  the rational functions

$$\delta_j(z) = \frac{r(z)}{r'(\lambda_j)(z - \lambda_j)} \tag{6}$$

we consider representations of scalar functions  $\varphi$  in the form

$$\varphi(z) = \sum_{j=1}^{d} \delta_j(z) f_j(w) \text{ where } w = r(z).$$
(7)



**Fig. 2** The rational function r(x) in (5) with |r(z)| = 5.6

The assumptions that the roots of p to be simple and q being of lower degree than p are not necessary but made for simplifying the discussion.

## 3.1 Existence and uniqueness

We are interested in using w = r(z) as a new complex variable and assume in the following that  $q(z) \neq 0$ . Modifying the discussion in section 2.1 in [7] we take  $w \in \mathbb{C}$  and denote by  $z_i = z_i(w)$  the *d* roots of

$$p(z) - w q(z) = 0.$$
 (8)

Let  $z_i(w_0)$  be a simple root. Then it is analytic at  $w = w_0$  with



**Fig. 3** Level curves of  $z \mapsto \frac{1}{2} |z - 1/z|$ 

$$z'_{j}(w_{0}) = \frac{q(z_{j}(w_{0}))}{p'(z_{j}(w_{0})) - w_{0} q'(z_{j}(w_{0}))}$$

Observe that since  $r' = \frac{1}{(p'-rq')}$ , all finite critical values of r agree with those of p - wq. So, let  $w_0 \neq \infty$  be a noncritical value of r(w) so that the roots  $z_j(w_0)$  are all distinct. Assuming that *the values of*  $\varphi$  *at these roots are all known*, we ask for "unknowns"  $f_k(w_0)$  satisfying the equations

$$\sum_{k=1}^{d} \delta_k(z_j(w_0)) f_k(w_0) = \varphi(z_j(w_0))$$
(9)

for j = 1, ..., d. We may write this as a linear system of equations

$$A(w_0)f(w_0) = \varphi(r^{-1}(w_0))$$
(10)

where  $A(w) = (\delta_k(z_j(w)))_{j,k}$  is a square matrix,  $f(w_0) \in \mathbb{C}^d$  has components  $f_k(w_0)$ and  $\varphi(r^{-1}(w_0)) \in \mathbb{C}^d$  has components  $\varphi(z_j(w_0))$ . At noncritical  $w_0$  the matrix  $A(w_0)$ is nonsingular, as is easily seen by rewriting (9) as

$$\sum_{k=1}^d \mathscr{C}_k(z_j) x_k = (q \; \varphi)(z_j)$$

where  $z_j = z_j(w_0)$ ,  $x_k = q(\lambda_k)f_k(w_0)$  and  $\ell_k$  denote the Lagrange interpolation polynomials at  $\lambda_k$ 

$$\ell_k(z) = \frac{p(z)}{p'(\lambda_k)(z - \lambda_k)}.$$

Assume now that  $M \subset \mathbb{C}$  is compact and let  $K = r^{-1}(M)$ . Denote by  $M_0 = M \setminus W_c$ where  $W_c$  denotes the set of critical values of r and put  $K_0 = r^{-1}(M_0)$ .

**Proposition 3.1** Given a function  $\varphi$  mapping  $K_0 \to \mathbb{C}$ , with  $K_0$  as above, there exists a unique f mapping  $M_0 \to \mathbb{C}^d$  such that

$$\varphi(z) = \sum_{k=1}^{d} \delta_k(z) f_k(r(z)) \tag{11}$$

holds for  $z \in K_0$ . The function f inherits the smoothness of  $\varphi$ . In particular, if  $\varphi$  is continuous or holomorphic in  $K_0$ , then f is continuous or holomorphic in  $M_0$ .

**Proof** Since A(w) is holomorphic in  $M_0$  and nonsingular, then so is  $A(w)^{-1}$ . The claims follow from

$$f(w) = A(w)^{-1} \varphi(r^{-1}(w)).$$

At a critical value  $w_c$  there are less equations and  $f_k(w_c)$ 's do exist but are not unique. It is therefore of interest to study what continuity conditions on  $\varphi$  guarantee continuity of  $f_k$ 's at critical values. We shall see, that if  $\varphi$  is holomorphic in K, then fcan be extended from  $M_0$  to M so that it is holomorphic also at the critical values. We shall discuss this using Cauchy integral. However, at this point it is natural to note, that the constant function  $\varphi : z \mapsto 1$  is represented by  $f : w \mapsto (1, ..., 1)^T$ .

**Lemma 3.2** Let deg  $q < \deg p$  and denote by  $Z_a$  the zeros of q. Assume  $z \notin Z_a$ . Then

$$\sum_{k=1}^{d} \delta_k(z) = 1.$$
 (12)

**Proof** For  $q(z) \neq 0$  we have  $\sum_{k=1}^{d} \delta_k(z) = \frac{1}{q(z)} \sum_{k=1}^{d} q(\lambda_k) \ell_k(z)$ . But the Lagrange interpolant of q equals q as deg  $q < \deg p$ .

### 3.2 Decomposing the Cauchy kernel

Assume again deg  $q < \deg p$  and consider

$$r[\lambda, z] = \frac{r(\lambda) - r(z)}{\lambda - z}.$$

Consider  $\lambda$  to be fixed and such that  $q(\lambda) \neq 0$ . Then  $z \mapsto q(z)r[\lambda, z]$  is a polynomial of degree d - 1. In fact, it is  $\mathcal{O}(z^{d-1})$  as  $z \to \infty$  while as  $z \to \lambda$  it tends to  $q(\lambda)r'(\lambda)$ . Hence the Lagrange interpolation gives

$$q(z)r[\lambda, z] = \sum_{j=1}^{d} \ell_j(z)q(\lambda_j)r[\lambda, \lambda_j].$$

But since  $r(\lambda_i) = 0$  we can rewrite this for  $q(z) \neq 0$  as

$$r[\lambda, z] = \sum_{j=1}^{d} \delta_j(z) \frac{r(\lambda)}{\lambda - \lambda_j}.$$
(13)

Hence we have the following representation for the Cauchy kernel.

**Proposition 3.3** Let r = p/q with deg  $q < \deg p$ . Then

$$\frac{1}{\lambda - z} = \sum_{j=1}^{d} \delta_j(z) K_j(\lambda, r(z))$$
(14)

where

$$K_j(\lambda, w) = \frac{1}{\lambda - \lambda_j} \frac{r(\lambda)}{r(\lambda) - w}$$

This allows us to conclude that if  $\varphi$  is holomorphic in  $\{z : |r(z)| < \rho\}$  and continuous in  $\{z : |r(z)| \le \rho\}$ , then  $f_j$  is holomorphic in  $|w| < \rho$ . To that end, denote by  $\gamma_\rho$  the contour with points along  $|r(\lambda)| = \rho$ , each finite curve oriented such that  $|r(z)| < \rho$  stays on the left hand side. We assume additionally *that*  $\gamma_\rho$  *contains no critical points of r*, making the components smooth. Denote

$$L_{\rho,j} = \frac{1}{2\pi} \int_{\gamma_{\rho}} \frac{|\mathrm{d}\lambda|}{|\lambda - \lambda_j|}$$

**Proposition 3.4** Suppose  $\varphi$  is holomorphic in  $\lambda$  for  $|r(\lambda)| < \rho$  and continuous in  $|r(\lambda)| \le \rho$ . For  $|w| < \rho$  then

$$f_{j}(w) = \frac{1}{2\pi i} \int_{\gamma_{\rho}} K_{j}(\lambda, w) \,\varphi(\lambda) \,\mathrm{d}\lambda \tag{15}$$

and  $f_j$  is holomorphic in  $|w| < \rho$  and can be expanded as a convergent series

$$f_j(w) = \sum_{k=0}^{\infty} \alpha_{j,k} w^k, \text{ where } |\alpha_{j,k}| \le L_{\rho,j} \rho^{-k-1} \max_{\lambda \in \gamma_{\rho}} |\varphi(\lambda)|.$$

**Proof** The contour  $\gamma_{\rho}$  consists of a finite number of smooth curves for which we have for z inside  $\gamma_{\rho}$ 

The claim follows substituting (14) into this.

Notice in particular that  $f_i$  is holomorphic at critical values  $|w_c| < \rho$ .

We can localize this representation inside any number of components of  $\gamma_{\rho}$ . In fact, let  $J \subset \{1, 2, ..., d\}$  and let  $\gamma_{\rho,J}$  consist of those components of  $\gamma_{\rho}$  which surround at least one  $\lambda_i$  with  $j \in J$ .

**Corollary 3.5** Assume  $\varphi$  is holomorphic inside  $\gamma_{\rho,J}$  and continuous up to  $\gamma_{\rho,J}$ . Then the previous proposition holds with  $\gamma_{\rho}$  replaced by  $\gamma_{\rho,J}$ . In particular, if  $j \notin J$ , then  $f_j = 0$  and for z inside  $\gamma_{\rho,J}$  we have

$$\varphi(z) = \sum_{j \in J} \delta_j(z) f_j(r(z))$$

**Proof** We may define  $\varphi = 0$  along the remaining components:  $\gamma_{\rho} \setminus \gamma_{\rho,J}$ .

#### 3.3 Derivative data at Λ

Denote by  $\partial$  the derivative w.r.t. z. Then we have (Lemma 4.1 in [7])

$$\varphi^{(\nu)} = \sum_{k=1}^d \sum_{\mu=0}^\nu \binom{\nu}{\mu} \delta_k^{(\nu-\mu)} \partial^\mu (f_k \circ r).$$

Proceeding as in the polynomial case, [7], it is easy to see that the formulas stay formally the same with r' in place of p'. Given the values  $\varphi^{(\nu)}(\lambda_j)$  we can compute  $f_i^{(\nu)}(0)$  from the following

$$r'(\lambda_j))^{\nu} f_j^{(\nu)}(0) = \varphi^{(\nu)}(\lambda_j) - h_{j,\nu}$$
(16)

where

$$h_{j,\nu} = \sum_{k=1}^{d} \sum_{\mu=0}^{\nu-1} {\nu \choose \mu} \delta_k^{(\nu-\mu)}(\lambda_j) \sum_{l=0}^{\mu} b_{\mu,l}(\lambda_j) f_k^{(l)}(0) + \sum_{l=0}^{\nu-1} b_{\nu,l}(\lambda_j) f_j^{(l)}(0).$$

Then the power series<sup>1</sup>

$$\sum_{\nu=0}^{\infty} \frac{f_j^{(\nu)}(0)}{\nu!} w^{\nu} \tag{17}$$

<sup>&</sup>lt;sup>1</sup> the last term is missing in the corresponding line in [7]

represents  $f_j$  in a disc with radius the same as the distance from origin to the closest singularity of  $f_j$ .

#### 3.4 Unital Banach algebra $C_{\Sigma}(M)$

A simple functional calculus for diagonalizable matrices can be defined via similarity transformation into diagonal form. If  $A = TDT^{-1}$  then one can define  $\varphi(A) = T\varphi(D)T^{-1}$  with  $\varphi(D) = \text{diag}(\varphi(d_i))$ . If A has nontrivial Jordan blocks, then the following is possible: take a "simplifying polynomial" p with critical points with matching multiplicities at the eigenvalues corresponding to the nontrivial Jordan blocks. Then p(A) is diagonalizable, and again,  $\varphi(A)$  is well defined via

$$\varphi(A) = \sum_{j=1}^{d} \ell_j(A) f_j(p(A)).$$

In [9] this was approached as follows. Consider the Banach space of continuous functions from a compact set M into  $\mathbb{C}^d$ , with max-norm. Then a "polyproduct" o was constructed such that if f represents  $\varphi$  and g represents  $\psi$  then f o g represents  $\varphi \psi$ :

$$(\varphi\psi)(z) = \sum_{j=1}^d \ell_j(z) (f \odot g)_j(p(z))$$

for  $z \in p^{-1}(M)$ .

We indicate the key steps as they go for the rational variable w = r(z) in the same way. To define the product, let  $\{e_i\}$  denote the standard basis of  $\mathbb{C}^d$ . At this point we assume we are given a  $d \times d$  multiplication table  $\Sigma = \{\sigma_{ii}\}$  and a frozen  $w \in M$ .

**Definition 3.6** Define in  $\mathbb{C}^d$ 

$$e_i \odot e_i = e_i - w \sum_{j \neq i} (\sigma_{ij} e_i + \sigma_{ji} e_j)$$

and for  $j \neq i$ 

$$e_i \odot e_j = w(\sigma_{ij}e_i + \sigma_{ji}e_j)$$

and extend to  $\mathbb{C}^d$  by linearity.

The product is clearly commutative. Denote  $\mathbf{1} = \sum_{i=1}^{d} e_i$ .

#### Lemma 3.7

$$\mathbf{1} \odot e_i = e_i$$
.

**Proof** We have using the definition

П

$$\mathbf{1} \odot e_i = \sum_j e_j \odot e_i = e_i \odot e_i + \sum_{j \neq i} e_j \odot e_i = e_i.$$

We shall now let  $w \in M$  to vary, with  $M \subset \mathbb{C}$  compact and write the functions  $f : M \to \mathbb{C}^d$  as

$$f: w \mapsto \sum_{j=1}^d f_j(w) e_j$$

and extend  $\odot$  to these functions in a natural way by treating  $f_j(w)$ 's as scalars. Passing to operator norm we obtain a *unital Banach algebra*. Denote as before,  $|f|_{\infty} = \max_{1 \le j \le d} \max_{w \in M} |f_j(w)|.$ 

**Proposition 3.8** Defining in  $C(M)^d$ 

$$\|f\| = \sup_{|g|_{\infty} \le 1} |f \odot g|_{\infty}$$

we have  $|f|_{\infty} \leq ||f|| \leq C|f|_{\infty}$  with C independent of f,  $||\mathbf{1}|| = 1$  and  $||f \odot g|| \leq ||f|| ||g||$ .

With this operator norm and polyproduct  $\odot$  generated by the table  $\Sigma$  we have a unital Banach algebra which we denote by  $C_{\Sigma}(M)$ .

**Proof** These properties hold in the similar way as in the polynomial case.  $\Box$ 

For the full power of Banach algebra machinery we need to know the set of characters. Recall that a continuous linear functional  $\phi: C_{\Sigma}(M) \to \mathbb{C}$ , is a *character* if it is nontrivial and multiplicative:

$$\phi(a \odot b) = \phi(a)\phi(b).$$

It is well known that all characters in C(M) are just evaluations  $\alpha \mapsto \alpha(w_0)$ , see e.g. [2]. We may identify the subalgebra of  $C_{\Sigma}(M)$  consisting of elements of the form  $w \mapsto \alpha(w)\mathbf{1}$  with C(M) and conclude that all characters in  $C_{\Sigma}(M)$  reduce to evaluations in this subalgebra. Fix  $w_0 \in M$  and let  $\chi$  be any character mapping  $\alpha \mathbf{1} \mapsto \alpha(w_0)$ . Then it follows that for any  $f \in C_{\Sigma}(M)$  the value  $\chi(f)$  only depends on  $f(w_0) \in \mathbb{C}^d$ . To see this, notice that for any f we have  $\alpha \mathbf{1} \odot f = \alpha f$  and so  $\chi(f) = \chi(\alpha f)$  provided  $\alpha(w_0) = 1$ . Let  $\alpha_n(w) = 1 - \min\{n \mid w - w_0\}, 1\}$ . Then  $\chi(f - f(w_0)) = 0$ . In fact, as  $\chi$ is continuous in the operator norm, which is equivalent with the max-norm,

$$|\chi(f - f(w_0))| = |\chi(\alpha_n(f - f(w_0))| \le C \|\chi\| |\alpha_n(f - f(w_0))|_{\infty} \to 0$$

as  $f - f(w_0)$  is continuous. Hence,  $\chi(f)$  only depends on the vector  $f(w_0) \in \mathbb{C}^d$ . Thus,  $\chi$  acts as evaluation at  $w_0$  followed by a multiplicative functional  $\mathbb{C}^d \to \mathbb{C}$  with  $\mathbb{C}^d$  equipped with the product O, where the variable *w* takes the fixed value  $w_0$ . But all linear functionals in  $\mathbb{C}^d$  are of the form

$$\eta : x = \sum_{i=1}^d \xi_i e_i \mapsto \sum_{i=1}^d \eta_i \xi_i.$$

Requiring  $\eta(1) = 1$  implies  $\sum_{i=1}^{d} \eta_i = 1$ . Consider first  $w_0 = 0$ . Then  $\eta(e_i \odot e_j) = 0$  implies that  $\eta$  has exactly one component size 1 while the others vanish. Thus there are *d* different characters. Let then  $w \neq 0$ . From

$$\eta(e_i \odot e_j) = \eta_i \eta_j = w_0[\sigma_{ij}\eta_i + \sigma_{ji}\eta_j]$$

we conclude that  $\eta_i \neq 0$  for all *i*. Applying to  $e_i \odot e_i$  we obtain

$$\eta_i^2 = \eta_i - w_0 \sum_{j \neq i} [\sigma_{ij} \eta_i + \sigma_{ji} \eta_j].$$
(18)

Taking e.g.  $\eta_i$  as an unknown, we can solve  $\eta_i$  for  $j \neq i$  from

$$\eta_i \eta_j = w_0 [\sigma_{ij} \eta_i + \sigma_{ji} \eta_j],$$

and substituting them into (18) yields a polynomial equation for  $\eta_i$  of degree *d*. Thus, again there are (at most) *d* characters for every  $w_0$ . In general, the components of characters depend on  $w_0$  in a rather complicated way. However, when the multiplication table  $\Sigma$  is given by a rational function, the dependence can be explicitly given.

**Definition 3.9** Let *p* be monic of degree *d* with simple roots  $\{\lambda_j\}$  and *q* of degree at most d - 1, with  $q(\lambda_j) \neq 0$  and denote r = p/q. If the multiplication table satisfies

$$\sigma_{ij} = \frac{1}{r'(\lambda_j)} \frac{1}{\lambda_i - \lambda_j},\tag{19}$$

then we say that the product  $\odot$  in  $C_{\Sigma}(M)$  is determined by the rational function r.

We shall next connect the products  $e_i \odot e_j$  to those of  $\delta_i \delta_j$ .

**Lemma 3.10** Assume that  $\Sigma$  is determined by the rational function r. Then

$$\delta_i^2 = \delta_i - \frac{p}{q} \sum_{j \neq i} \left[ \sigma_{ij} \delta_i + \sigma_{ji} \delta_j \right], \text{ while for } i \neq j, \ \delta_i \delta_j = \frac{p}{q} \left[ \sigma_{ij} \delta_i + \sigma_{ji} \delta_j \right]. \tag{20}$$

**Proof** In the polynomial case with q = 1 this is Lemma 1 in [9]. In fact, we have in the polynomial case

$$\ell_i^2 = \ell_i - p \sum_{j \neq i} \left[ \tau_{ij} \ell_i + \tau_{ji} \ell_j \right], \text{ while for } i \neq j, \ \ell_i \ell_j = p \left[ \tau_{ij} \ell_i + \tau_{ji} \ell_j \right], \quad (21)$$

where  $\tau_{ij} = \frac{1}{p'(\lambda_i)} \frac{1}{\lambda_i - \lambda_j}$ . Since  $\delta_i = \frac{q(\lambda_i)}{q} \ell_i$ ,  $\sigma_{ij} = q(\lambda_j) \tau_{ij}$ , and  $q = \sum_{i=1}^d q(\lambda_i) \ell_i$ , (20) follows from (21). Assume now that w is not critical, so that there are d different roots  $z_j = z_j(w)$  satisfying p(z) - wq(z) = 0. So, for each of these roots we have, with f and g representing  $\varphi$  and  $\psi$  respectively, that

$$(\varphi \psi)(z_j(w)) = \sum_{i=1}^d \delta_i(z_j)(f \odot g)_i(w),$$

which means that for  $z \in r^{-1}\{w\}$  and  $w \in M$ 

$$\chi_{z} : f \mapsto \sum_{i=1}^{d} \delta_{i}(z) f_{i}(w)$$
(22)

is a character. Observe that the character first evaluates f at a point followed by application by functional in  $\mathbb{C}^d$  with components  $\delta_i(z)$ . As different roots  $z_j(w)$  give d different characters, all satisfying the polynomial equation for the components of  $\eta$  discussed above, we conclude that we have found all characters. We may summarize this in the following.

**Theorem 3.11** In the unital commutative Banach algebra  $C_{\Sigma}(M)$  with  $\Sigma$  generated by a rational function as in Definition 3.9 all characters are of the form (22). In particular the Gelfand transformation  $f \mapsto \hat{f}$  is given by  $\hat{f}(z) = \chi_z(f)$ .

Since  $\varphi(z) = \chi_z(f)$  the spectrum of f is simply  $\sigma(f) = \{\varphi(z) : z \in r^{-1}(M)\}$ . In [9] the polynomial case is analysed in detail. For example, if  $|\varphi(z)| \ge \eta > 0$  in  $K = p^{-1}(M)$ , then there exists  $g \in C_\lambda(M)$  such that  $g \odot f = \mathbf{1}$  satisfying

$$||g|| \le C \frac{||f||^{d-1}}{n^d},$$

where the constant *C* only depends on *M* and on *p*. Further, when applying the functional calculus the set *M* must contain  $\sigma(p(A))$ , but then the inverse image  $K = p^{-1}(M)$  may be essentially larger than  $\sigma(A)$ . In such case a quotient algebra appears useful.

**Example 3.12** As a simple rational function which is not a Möbius transformation, consider

$$r(z) = \frac{1}{2} \left( z - \frac{1}{z} \right). \tag{23}$$

We have  $Z_p = \{1, -1\}, Z_q = \{0\}, r'(1) = r'(-1) = 1$ , and thus

$$\delta_1(z) = \frac{1}{2} \left( 1 + \frac{1}{z} \right), \ \delta_2(z) = \frac{1}{2} \left( 1 - \frac{1}{z} \right), \ \sum_1^2 \delta_i(z) = 1.$$

Further, with  $\sigma_{i,j} = \frac{1}{r'(\lambda_j)} \frac{1}{\lambda_i - \lambda_j}$  we have  $\sigma_{1,2} = \frac{1}{2}$ ,  $\sigma_{2,1} = -\frac{1}{2}$ . From  $z^2 - 2wz - 1 = 0$ , we obtain the inverse images  $z_{\pm}(w) = w \pm \sqrt{1 + w^2}$ . The critical points are at  $z_c = \pm i$ , with critical values  $w_c = \pm i$ .

The matrix  $A(w) = (\delta_i(z_i(w)))$  mapping f to  $\varphi$  in (10) is

$$A(w) = \frac{1}{2\sqrt{1+w^2}} \begin{pmatrix} 1+\sqrt{1+w^2}-w \ 1-\sqrt{1+w^2}+w\\ 1-\sqrt{1+w^2}-w \ 1+\sqrt{1+w^2}+w \end{pmatrix}$$
(24)

with inverse

$$A(w)^{-1} = \frac{1}{2\sqrt{1+w^2}} \begin{pmatrix} 1+\sqrt{1+w^2}+w & -1+\sqrt{1+w^2}-w\\ -1+\sqrt{1+w^2}+w & 1+\sqrt{1+w^2}-w \end{pmatrix}.$$
 (25)

For example, the variable *z* is represented by

$$f(w) = A(w)^{-1} \begin{pmatrix} z_+(w) \\ z_-(w) \end{pmatrix} = \begin{pmatrix} 1+2w \\ -1+2w \end{pmatrix}.$$

#### 3.5 Relation between representations using p and r

As we may take both p and r = p/q as new variables there naturally exists a mapping between the representations. In fact, let  $\varphi$  be holomorphic in all that comes and suppose the multicentric representation using polynomial variable w = p(z) is denoted as

$$\varphi(z) = \sum_{j=1}^{d} \ell_j(z) F_j(p(z))$$

and

$$q(z) = \sum_{j=1}^{d} \ell_j(z) Q_j(p(z))$$

where  $Q_i(w) = q(\lambda_i)$ . Then

$$(\varphi q)(z) = \sum_{j=1}^{d} \ell_j(z) (F \odot Q)_j(p(z))$$
(26)

Using r = p/q as the new variable we have

$$\varphi(z) = \sum_{j=1}^{d} \delta_j(z) f_j(r(z))$$

from which we obtain

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$$(\varphi q)(z) = \sum_{j=1}^{d} \ell_j(z) q(\lambda_j) f_j(r(z)).$$
(27)

Hence we have

$$(F \odot Q)(p(z)) = (f \circ Q) \left(\frac{p}{q}(z)\right), \tag{28}$$

where we denote by o the elementwise product.

Finally, if we write  $Q^{-1}$  for the vector with components  $1/q(\lambda_i)$  we have

$$f(\frac{p}{q}(z)) = \left(Q^{-1} \circ F \odot Q\right)(p(z))$$

## 4 Application to Sylvester equation

Let *A* and *B* be bounded operators in Banach spaces  $\mathcal{X}$ ,  $\mathcal{Y}$  respectively. Then AX - XB = C is the related Sylvester equation, where *C* is a given operator and *X* the unknown, both mapping  $\mathcal{Y}$  to  $\mathcal{X}$ . It is well known [3] that a unique bounded *X* exists for every bounded *C* if and only if *A* and *B* have disjoint spectra:  $\sigma(A) \cap \sigma(B) = \emptyset$ . In Section 5 of [10] the multicentric calculus was applied to the case where the polynomial convex hulls of the spectra were disjoint:  $\sigma(A) \cap \sigma(B) = \emptyset$ . The solution was constructed as a convergent power series provided that one has a polynomial lemniscate which separates the spectra into different components. Here we outline the approach using rational lemniscates, which then removes the need to assume that the polynomially convex hulls do not intersect.

Denote

$$M = \begin{pmatrix} A & C \\ B \end{pmatrix}.$$
 (29)

Observe that

$$M = \begin{pmatrix} I & -X \\ I \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} I & X \\ I \end{pmatrix}$$
(30)

is satisfied exactly when AX - XB = C. Denote by sgn(z) the function taking value 1 in the open right half plane  $\mathbb{C}_+$  and -1 in the left one. If  $\sigma(A) \subset \mathbb{C}_+$  and  $\sigma(B) \subset \mathbb{C}_-$ , then sgn(M) is well defined and we have

$$\operatorname{sgn}(M) = \begin{pmatrix} I & -X \\ I \end{pmatrix} \begin{pmatrix} I \\ -I \end{pmatrix} \begin{pmatrix} I & X \\ I \end{pmatrix} = \begin{pmatrix} I & 2X \\ -I \end{pmatrix}.$$
 (31)

Thus, *X* can be obtained if sgn(M) can be computed, see e.g. [3, 6].

$$Q = \frac{1}{2\pi i} \int_{\gamma_1} (\lambda - M)^{-1}$$
(32)

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we have

$$Q = \begin{pmatrix} I & -X \\ I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & X \\ I \end{pmatrix} = \begin{pmatrix} I & X \\ 0 \end{pmatrix}.$$

Now define, in place of the sign-function,  $\psi(z) = 1$  for  $z \in U_1$  while  $\psi(z) = -1$  for  $z \in U_2$ . In order to have a convergent series expansion for X let  $\gamma_2$  be a contour surrounding  $\sigma(B)$  inside  $U_2$  and denote  $\gamma = \gamma_1 \cup \gamma_2$ . Thus

$$I = \frac{1}{2\pi i} \int_{\gamma} (\lambda - M)^{-1}$$

But then adding this to both sides of

$$\psi(M) = Q - \frac{1}{2\pi i} \int_{\gamma_2} (\lambda - M)^{-1}$$

yields  $\psi(M) = 2Q - I$  and  $Q = \frac{1}{2}(\psi(M) + I)$ .

tour surrounding  $\sigma(A)$  inside  $U_1$ . Then denoting

Hence we have reduced the solving of the Sylvester equation into computing  $\psi(M)$ . In order to do that we need a rational function which separates the spectra of A and B.

**Proposition 4.1** Let M in (29) be given and such that  $\sigma(A) \cap \sigma(B) = \emptyset$ . Let  $\varepsilon > 0$  satisfy  $\varepsilon < \text{dist}(\sigma(A), \sigma(B))/2$ . Then there exists a rational function r = p/q such that p has distinct roots, deg q < deg p and such that the components of the lemniscate set

$$V = \{z : |r(z)| < 1\}$$
(33)

can be grouped into three disjoint sets:  $V = V_1 \cup V_2 \cup V_0$  where  $\sigma(A) \in V_1$ ,  $\sigma(B) \in V_2$ , while  $V_0$ , which may empty, satisfies  $V_0 \cap \sigma(M) = \emptyset$ . Further, dist $(V_i, V_j) \ge \varepsilon/2$  for  $i \ne j$ .

**Proof** This follows from Corollary 2.2. by applying it to the operator M. In fact, there exists a rational function r such that ||r(M)|| < 1 while for  $z \in \Gamma_{\varepsilon}$  we have |r(z)| > 1. We have

$$\sigma(M) = \sigma(A) \cup \sigma(B) \subset \{z : |r(z)| \le ||r(M)||\} \Subset V.$$

Then collect all components of V for which the distance to  $\sigma(A)$  is at most  $\varepsilon/2$  into  $V_1$ , those which are likewise close to  $\sigma(B)$  into  $V_2$  and the rest, if any, into  $V_0$ . Now  $V_1$  and  $V_2$  are surrounded by  $\Gamma_{\varepsilon}$  of width  $\varepsilon/2$ , and the claims follow.

Assume now that *r* satisfies the assumptions of the previous proposition. We set  $\psi(z) = 1$  in  $V_1$ ,  $\psi(z) = -1$  in  $V_2$  and  $\psi(z) = 0$  in  $V_0$ . Then we have

$$\psi(z) = \sum_{j=1}^d \delta_j(z) h_j(r(z))$$

where  $h_i$ 's are holomorphic for |w| < 1. Computing the power series  $h_j(w) = \sum_{k=0}^{\infty} h_{j,k} w^k$  then gives an explicit expression for  $\psi(M)$ .

**Proposition 4.2** Under the notation and assumptions above, the upper right corner element of  $\psi(M)$  is 2X where X is the solution of the Sylvester equation AX - XB = C.

Notice that the power series converges for |w| < 1. When ||r(M)|| < 1 one can truncate the power series with the possibility to bound the truncation error. Notice that the asymptotic convergence factor  $\eta$  is given by

$$\eta = \max_{\lambda \in \sigma(M)} |r(\lambda)|$$

and is hence independent of the "right hand side" C.

**Example 4.3** Let again r(z) = (z - 1/z)/2 and  $M = \begin{pmatrix} a & c \\ b \end{pmatrix}$  be such that  $\sqrt{2} - 1 < a, -b < \sqrt{2} + 1$ , so that |r(a)|, |r(b)| < 1. Hence, this serves as miniature model for solving the "Sylvester equation"  $a\xi - \xi b = c$  along the lines above. The set where |r(z)| < 1 has two components, one in the right half plane and the other in the left. Thus choosing  $\psi$  to take the value 1 in the component with Re z > 0 and -1 in the other one we actually arrive into the restriction of sign-function into these sets. This however just follows from the simple form of r. So, the answer shall be

$$\psi(M) = \begin{pmatrix} 1 & \frac{2c}{a-b} \\ & -1 \end{pmatrix}$$

with  $2\xi$  appearing in the upper right hand corner but we proceed without knowing the simple answer. Thus, we need to have *f* representing this  $\psi$  in the unit circle |w| < 1 and this is given immediately from (25)

$$f(w) = A(w)^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{1+w^2}} \begin{pmatrix} 1+w \\ -1+w \end{pmatrix}.$$

Hence for z in |r(z)| < 1 we have  $\psi(z) = \delta_1(z)f_1(r(z)) + \delta_2(z)f_2(r(z))$  which simplifies into

$$\psi(z) = t(z)(t(z)^2)^{-1/2}$$
, where  $t(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ . (34)

Since  $t(z)^2 = 1 + r(z)^2$  we arrive to an explicit series expansion for  $\psi$ :

$$\psi(z) = t(z) \left( 1 - \frac{1}{2}r(z)^2 + \frac{3}{8}r(z)^4 - \cdots \right).$$
(35)

Notice that if  $|a-1|, |b+1| \le \varepsilon$ , then the spectral radius of  $r(M)^2$  satisfies  $\rho(r(M)^2) < \varepsilon^2$  and the convergence of the series for  $(t(M)^2)^{-1/2}$  would be rapid and truncation could be done safely. The situation would remain similar if the scalars *a* and *b* would be replaced with bounded operators *A* and *B* with spectra near 1 and -1, respectively.

## 5 Application to K-spectral sets

It was shown in [8] that polynomial lemniscate sets are K-spectral sets, provided that the boundaries are smooth, i.e. do not contain critical points. Here we point out that this extends to rational lemniscates. The proof in [8] goes as follows. Representing the holomorphic function in the multicentric form leads us to estimate the components of *f* evaluated at p(A). But since *f* maps in a disc, we can apply the von Neumann inequality to get  $||f_j(p(A))|| \le |f_j|_{\infty}$ . The third step needed, is to bound *f* in terms of  $\varphi$ . To repeat this in the rational lemniscate case, we formulate the last step in the following lemma.

**Lemma 5.1** Suppose r = p/q where p has simple roots  $\lambda_j$ ,  $q(\lambda_j) \neq 0$  and deg  $q < \deg p = d$ . Suppose R is such that |r(z)| = R contains no critical points of r. Then there exists a constant C(r, R) such that for all f with components  $f_j$  holomorphic in  $|w| \leq R$  there holds

$$\sup_{|w| \le R} |f(w)|_{\infty} \le C(r, R) \sup_{|r(z)| \le R} |\varphi(z)|.$$

**Proof** This follows from the Cauchy integral formulation.

Then we have the following.

**Theorem 5.2** Let r and R be as in the previous lemma. Suppose A is a bounded operator in a Hilbert space such that  $||r(A)|| \le R$ . If  $\varphi$  is holomorphic in  $V_r(R) = \{z : |r(z)| \le R\}$ , then

$$\|\varphi(A)\| \le K \sup_{V_r(R)} |\varphi| \tag{36}$$

where  $K = C(r, R) \sum_{j=1}^{d} \|\delta_j(A)\|$ .

Proof In

$$\|\varphi(A)\| \le \sum_{j=1}^d \|\delta_j(A)\| \|f_j(r(A))\|$$

we apply the von Neumann inequality to get  $||f_j(r(A))|| \le \sup_{|w|\le R} |f_j(w)|$  and then bound these by  $\sup_{V_r(R)} |\varphi|$  using Lemma 5.1.

Thus, the sets are K-spectral sets with constant independent of the holomorphic function  $\varphi$  but depending on the geometry of the set and on A through  $\delta_i(A)$ .

**Remark 5.3** The constant C(r, R) depends on the distance from the lemniscate to critical points ns ia independent of the operator A. In [8] it is shown that in the polynomial case we have

$$C(p,R) \le 1 + \frac{C}{s(R)^{d-1}}$$
 (37)

where s(R) denotes the distance to the nearest critical point. Generically the behavior is proportional to 1/s(R) but higher powers occur with possible multiplicities of the critical points. Example 2.4 in [8] shows that the worst case in (37) can happen. Recall that we denoted by A(w) the matrix mapping f to  $\varphi$ , see (10). If R is small enough so that all critical values  $w_c$  satisfy  $|w_c| > R$ , then  $A(w)^{-1}$  is holomorphic for  $|w| \le R$  and we have

$$C(r, R) \le \sup_{|w| \le R} ||A(w)^{-1}||_{\infty}$$

where we denote by  $\|\cdot\|_{\infty}$  the matrix norm induced by the max-norm in  $\mathbb{C}^d$ . As the growth exponent in C(r, R) when  $s(R) \to 0$  depends on the multiplicity of the critical points and this behavior is local in nature, we shall not repeat the argument as it goes in the same way as in the polynomial case. Rather, we again return to the simple d = 2 case with r(z) = (z - 1/z)/2.

**Example 5.4** The mapping matrix A(w) for r(z) = (z - 1/z)/2 has the inverse given in (25). When  $R = 1 - \varepsilon$  and  $\varepsilon \to 0$  the distance s(R) behaves like  $(1 + o(1)) \varepsilon$  and

$$||A(w)^{-1}||_{\infty} = 1 + (1 + o(1)) \varepsilon^{-1}.$$

**Example 5.5** The other consider again the rational function r(z) = (z - 1/z)/2 together with the matrix

$$A = \begin{pmatrix} 1 & c \\ 0 & -1 \end{pmatrix}.$$

Since r(A) = 0 the coefficient C(r, R) in Lemma 5.1 shrinks to C(r, 0) = 1 and (36) holds with  $K = \sum_{1}^{2} ||\delta_{j}(A)||$ . With  $\delta_{1}(z) = (1 + 1/z)/2$  and  $\delta_{2}(z) = (1 - 1/z)/2$  we have

$$\delta_1(A) = \begin{pmatrix} 1 & c/2 \\ 0 & 0 \end{pmatrix}, \ \delta_2(A) = \begin{pmatrix} 0 & -c/2 \\ 0 & 1 \end{pmatrix}.$$

For example, the Riesz projection with respect to the eigenvalue  $\lambda_1 = 1$  is  $\delta_1(A)$  while with  $\varphi(z) = z$  we have

$$A = \delta_1(A) \cdot 1 + \delta_2(A) \cdot (-1)$$

which shows that  $K = \sum_{1}^{2} \|\delta_{j}(A)\|$  becomes tight when |c| grows. Finally, notice that *A* is of the form

$$M = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$$

for which the corresponding "Sylvester equation" reads ax - xb = c with solution x = c/(a - b) = c/2 to be found in the upper right corners of  $\delta_1(A)$  and  $-\delta_2(A)$ .

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## References

- Andrei, D.: Multicentric holomorphic calculus for n-tuples of commuting operators. Adv. Oper. Theory 4(2), 447–461 (2019)
- 2. Aupetit, B.: A Primer on Spectral Theory. Springer, Berlin (1991)
- Bhatia, R., Rosenthal, P.: How and why to solve the operator equation AX XB = Y. Bull. Lond. Math. Soc. 29, 1–21 (1997)
- Diana Apetrei, O.: Nevanlinna. Multicentric calculus and the Riesz projection. J. Numer. Anal. Approx. Theory 44(2), 127–145 (2016)
- 5. Gaier, D.: Lectures on Complex Approximation. Birkhäuser, Basel (1985)
- Higham, N.J.: Functions of Matrices. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2008)
- Nevanlinna, O.: Multicentric holomorphic calculus. Comput. Methods Funct. Theory 12(1), 45–65 (2012)
- 8. Nevanlinna, O.: Lemniscates and K-spectral sets. J. Funct. Anal. 262, 1728–1741 (2012)
- Nevanlinna, O.: Polynomial as a new variable—a Banach algebra with functional calculus. Oper. Matrices 10(3), 567–592 (2016)
- Nevanlinna, O.: Sylvester equations and polynomial separation of spectra. Oper. Matrices 13(3), 867–885 (2019)
- 11. Ransford, T.: Potential Theory in the Complex Plane, London Math. Soc. Student Texts 28, Cambridge University Press, Cambridge (1995)