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Algorithms and complexity for counting configurations in Steiner triple systems

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Abstract

Steiner triple systems form one of the most studied classes of combinatorial designs. Configurations, including subsystems, play a central role in the investigation of Steiner triple systems. With sporadic instances of small systems, ad hoc algorithms for counting or listing configurations are typically fast enough for practical needs, but with many systems or large systems, the relevance of computational complexity and algorithms of low complexity is highlighted. General theoretical results as well as specific practical algorithms for important configurations are presented.

K E Y W O R D S

algorithm, computational complexity, configuration, Steiner triple system, subsystem

MATHEMATICAL SUBJECT CLASSIFICATION 05B07, 68Q25

1 | INTRODUCTION

A *Steiner triple system* (STS) is an ordered pair (V, \mathcal{B}) , where *V* is a set of *points* and \mathcal{B} is a set of 3-subsets of points, called *blocks* or *lines*, such that every 2-subset of points occurs in exactly one block. The size of the point set is the *order* of the Steiner triple system, and a Steiner triple system of order v is denoted by STS(v). It is well known that an STS(v) exists iff

$$v \equiv 1 \text{ or } 3 \pmod{6}.$$

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An STS(ν) has $\nu(\nu - 1)/6$ blocks and each point is in $(\nu - 1)/2$ blocks. For more information about Steiner triple systems, see [5,9].

A configuration in a (V, \mathcal{B}) STS(v) is a set system (V', \mathcal{B}') , where $\mathcal{B}' \subseteq \mathcal{B}$ and $V' = \bigcup_{B \in \mathcal{B}'} B$. For configurations, we adopt the convention of calling the elements of \mathcal{B}' lines. If each point in V' occurs in at least two lines, then the configuration is said to be *full*. A configuration that is an STS(w) is called an STS(w) subsystem, or a sub-STS(w), and is said to be proper if w < v and nontrivial if w > 3. A configuration with w lines such that each point is in three of the lines is a w_3 configuration [16]. Double counting shows that the size of the point set of a w_3 configuration is w.

The computational problem of finding configurations in designs is recurrent in design theory. For example, the problem of finding maximal arcs in projective planes of order 16, studied in [13], is about finding 2-(52, 4, 1) subdesigns in 2-(273, 17, 1) designs. Similar computational problems also occur in discrete geometry [3]. We shall here focus explicitly on configurations in Steiner triple systems. This work is motivated by a need in various studies for algorithms to count configurations in many Steiner triple systems with orders that are large. Earlier studies in this area have mainly concerned subsystems of Steiner triple systems [7].

An STS(v) is said to be *isomorphic* to another STS(v) if there exists a bijection between the point sets that maps blocks onto blocks; such a bijection is called an *isomorphism*. An isomorphism of a Steiner triple system onto itself is an *automorphism* of the Steiner triple system. The automorphisms of a Steiner triple system form a group under composition, the *automorphism group* of the Steiner triple system. These concepts are defined analogously for configurations.

The paper is organized as follows. In Section 2, the time complexity of counting and listing configurations in Steiner triple systems is considered. With fixed (sets of) configurations, these problems are in **P**. Polynomial upper bounds on the time complexity are obtained by developing algorithms. The number of occurrences of an *n*-line configurations with $m \le n$. The conjecture that a subset of the full *m*-line configurations does not suffice has earlier been verified for $n \le 7$, which is here extended to $n \le 8$. Practical aspects are not addressed in the theoretical proofs, so Section 3 is devoted to practical counting algorithms for several specific small configurations. In particular, an approach is developed for constructing an exhaustive set of algorithms of a certain type. The algorithms are compared experimentally, and the winning algorithms are displayed for nine important configurations.

2 | COUNTING CONFIGURATIONS

2.1 | Problem and algorithms

The computational problem studied here is as follows, where C is a set of configurations. Throughout the paper we consider C and any related parameters to be *fixed* and *finite*. (The problem could also have been defined with |C| = 1, but it is more natural to consider a larger set as one often asks questions regarding classes of configurations.)

Problem: P(C).

Input: A Steiner triple system S of order v.

Output: The number of occurrences of the configurations in S that are isomorphic to a configuration in C.

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The motivation for the work is that of *counting* configurations, but we will also address the problem of *listing* configurations in Steiner triple systems. The listed configurations can obviously simultaneously be counted, but the fastest (known) counting algorithm has in many cases smaller time complexity than the fastest (known) listing algorithm. For example, considering a configuration with just one line, an optimal algorithm simply lists all lines of the STS(v), which takes $\Theta(v^2)$ time, whereas counting is simply a matter of evaluating v(v - 1)/6.

Indeed, for certain configurations the number of occurrences in an STS(v) only depends on v. Such configurations are called *constant*. A configuration that is not constant is said to be *variable*. Small configurations in STS(v)s are surveyed in [9, Chap. 13]. All configurations with three or fewer lines are constant and so are the members of five infinite families presented in [19]. A complete characterization of constant configurations is still missing.

The fact that the set of configurations C is fixed and finite gives possibilities of simplifying proofs. For example, determining whether two configurations are isomorphic can then be done in O(1) time, that is, constant time. As the goal is to establish theoretical bounds, we do not make any attempts to develop practical algorithms in this section but defer such issues to Section 3.

It is straightforward to see that Problem P(C) is in **P**. Namely, if there are at most *m* lines in the configurations in C, then we can explore all subsets of at most *m* lines of S and there are $\Theta(v^{2m})$ such subsets. However, it turns out that the upper bound thereby obtained is weak, and better upper bounds—also in the context of listing—can be obtained. As **P** \subseteq **PSPACE**, the space complexity will also be polynomial in all cases and will not be considered here.

The concept of configurations generated by sets of points is central to the study of specific algorithms. We here use a framework considered for Steiner triple systems in [9, p. 99] and its references. Given a configuration with point set *V* and line set \mathcal{B} , fix $V_0 \subseteq V$ and let

$$V_{i+1} = V_i \cup \{z : \{x, y, z\} \in \mathcal{B}, x, y \in V_i\}.$$

We further denote

$$\mathcal{B}(V_i) \coloneqq \{\{x, y, z\} \in \mathcal{B} : \{x, y, z\} \subseteq V_i\}.$$

For some finite j, $V_{j+1} = V_j$ and then $V_i = V_j$ for all i > j. The subconfiguration $(V_j, \mathcal{B}(V_j))$ is called the *closure* of V_0 and is said to be *generated* by V_0 .

If $|V_0| = 2$, then V_0 generates a subconfiguration that contains at most one line of \mathcal{B} , a line in which the pair of points occurs. Similarly, if $|V_0| = 3$ and V_0 is a line of \mathcal{B} , then the subconfiguration generated by V_0 consists of just the line V_0 . But larger subconfigurations can be generated by V_0 when $|V_0| = 3$ and V_0 is not a line of \mathcal{B} and when $|V_0| > 3$. Cases where V_0 generates the entire configuration (V, \mathcal{B}) are of particular interest.

Theorem 1. Let C be a set of configurations that can be generated by m points. Then the time complexity of listing the configurations in an STS(v) isomorphic to a configuration in C is $O(v^m)$.

Proof. Consider a (V, \mathcal{B}) STS(v). For each possible *m*-subset $V'_0 \subseteq V$ and with $\mathcal{B}'_0 = \emptyset$, the following iterative extension procedure is carried out in all possible ways: given a point set V'_i and a line set \mathcal{B}'_i , let $V'_{i+1} = V'_i \cup B$ and $\mathcal{B}'_{i+1} = \mathcal{B}'_i \cup \{B\}$, where $B \in \mathcal{B} \setminus \mathcal{B}'_i$ and $|B \cap V'_i| \ge 2$. Whenever *i* equals the number of lines of a configuration in \mathcal{C} , an isomorphism test is carried out. If the outcome of that test is positive, then the

configuration is listed if two additional tests are passed: (i) the *m*-subset V'_0 is the lexicographically smallest one amongst the *m*-subsets from which the configuration can be generated, and (ii) the configuration has not already been listed in the branch of the search tree starting from V'_0 . This makes sure that each configuration is listed exactly once. The number of lines in the configurations in C sets a bound on the largest value of *i* to consider.

Using a precomputed data structure, which can be created in $O(v^2)$ time (cf. Section 3), the extension can be carried out in constant time. As the isomorphism test can also be carried out in constant time, the time complexity of the problem is bounded from above by the number of *m*-subsets of a *v*-set and is therefore $O(v^m)$.

Note that the core of the algorithm in the proof is essentially about canonical augmentation [22]—see also [20, Sect. 4.2.3]—which consists of (i) a parent test and (ii) an isomorphism test. The extension procedures in and before the proof of Theorem 1 are closely connected to the core of Miller's algorithm [23] for computing a canonical form of an STS(ν) in $O(\nu^{\log \nu + O(1)})$ time. Further related studies include [7,25]. Also note the similarity between the extension procedure and the algorithm in [3].

For listing algorithms, it is now a matter of determining the size of point sets needed to generate configurations.

2.2 | Small configurations

An algorithm that solves Problem P(C) gives an upper bound on the time complexity. For some configurations, including the smallest nontrivial case of *Pasch* configurations, it is possible to prove that the upper bound given by Theorem 1 is actually exact. The Pasch configuration is depicted in Figure 1A; one possible set of generating points is here and in later pictures shown with bold circles. The labels in all pictures refer to the naming of variables in Section 3.

Theorem 2. The time complexity of listing the Pasch configurations in an STS(v) is $\Theta(v^3)$.

Proof. The Pasch configuration can be generated by the three points indicated in Figure 1A. Then combine the upper bound given by Theorem 1 and the fact that there are STS(v)s with v(v - 1)(v - 3)/24 Pasch configurations [26].



FIGURE 1 The Pasch and mitre configurations

Theorem 3. The time complexity of counting the number of occurrences of a given 4-line variable configuration in an STS(v) is $O(v^3)$.

Proof. For any of the 11 variable 4-line configurations, the number of occurrences can be derived from the number of occurrences of any single one of them [15]. In particular, using the number of Pasch configurations, the result follows from Theorem 2. \Box

The case of 4-line configurations provides further examples with different time complexities for counting and listing. Namely, for all but the Pasch configuration, 4 to 8 points are needed for generation, which gives listing algorithms with time complexity $O(v^4)$ to $O(v^8)$. In fact, the maximum number of occurrences of configurations [15] shows that the time complexity of listing is $\Theta(v^4)$ to $\Theta(v^8)$. For example, the configuration that requires eight points for generation consists of four disjoint lines.

The result in Theorem 3 may be extended by considering *n*-line configurations for any fixed n > 4 in an analogous way. Formulas for the relationship between the numbers of occurrences of variable 5-line and 6-line configurations can be found in [10] and [11], respectively. Actually, since we do not need exact formulas in our study of complexity, the following general result will suffice.

Theorem 4 (Horak et al. [19]). The number of occurrences of any variable *n*-line configuration in an STS(v) is a polynomial in v plus a linear combination (with coefficients that are polynomials in v) of the numbers of occurrences of the full *m*-line configurations with $m \le n$.

The only full 5-line configuration is the *mitre* configuration (Figure 1B).

Theorem 5. The time complexity of counting the number of occurrences of a given 5-line variable configuration in an STS(v) is $O(v^3)$.

Proof. The full *m*-line configurations with $m \le 5$ are the Pasch and the mitre configuration. The mitre configuration can be generated by three points—as indicated in Figure 1B—and therefore the number of occurrences can be obtained in $O(v^3)$ time by Theorem 1. As also the Pasch configurations can be counted in $O(v^3)$ time by Theorem 3, the result now follows from Theorem 4.

With an increasing number of lines, extending the results in Theorems 3 and 5 is rather straightforward but becomes more and more laborious. Moreover, for different configurations we will get different upper bounds on the time complexity, so the results cannot be stated in compact form. For example, for 6-line and 7-line configurations we get the following general result.

Theorem 6. The time complexity of counting the number of occurrences of a given 6-line or 7-line variable configuration in an STS(v) is $O(v^4)$.

Proof. There are five full 6-line configurations, which are depicted in Figure 2 with generating sets indicated. Consequently, by Theorem 4, the time complexity for counting the number of occurrences of a variable 6-line configuration is $O(v^4)$. For variable 7-line



FIGURE 2 The full 6-line configurations

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configurations, a similar argument applies as all full 7-line configurations have generating sets of size at most 4 (by Table 1, to be discussed later). \square

Clearly Theorem 6 is not tight in the sense that for some of the variable 6-line and 7-line configurations, the time complexity of counting is $O(v^3)$.

For a fixed $n \ge 7$, there are too many full *n*-line configurations to depict all of them here. However, computationally one can easily get rather extensive results. The number N of isomorphism classes of full *n*-line configurations for $n \leq 12$ have been obtained earlier in [12]. We extend that work to $n \leq 13$ in Table 1 and, for all those parameters, tabulate the distribution N_i based on the size *i* of the smallest generating set. Also, the distribution of automorphism group sizes, |Aut|, is shown such that the bases and exponents give the group sizes and counts, respectively.

It is an interesting open question whether all full configurations are really required in Theorem 4 or whether a subset of them would suffice. It is conjectured in [19] that Theorem 4 is indeed strict. For n = 4, 5, and 6, this follows from the formulas of [15], [10], and [11], respectively, and the case of n = 6 is handled explicitly in [19]. The conjecture has also been verified for n = 7 in an unpublished study [27].

An established approach [14] to study the aforementioned conjecture for small values of *n* is to investigate the number of occurrences of full *m*-line configurations for $m \le n$ in a set of STS(v) s for some fixed v. If there are r configurations to consider, we get for each STS(v) a vector of length r + 1 with nonnegative integers (r counts and a constant, say 1). Forming a matrix with rows consisting of those vectors, we check whether the rank is r + 1 (for which we obviously need at least r + 1 vectors).

n	N	Aut	N_3	N_4	N_5	N_6	N_7	N_8	<i>N</i> ₉
4	1	24 ¹	1	0	0	0	0	0	0
5	1	12 ¹	1	0	0	0	0	0	0
6	5	$2^{1}12^{2}24^{1}72^{1}$	3	2	0	0	0	0	0
7	19	$1^3 2^5 4^3 6^5 12^2 168^1$	13	6	0	0	0	0	0
8	153	$\frac{1^{58}2^{50}3^14^{22}6^28^612^316^524^1}{32^148^264^11152^1}$	98	48	6	1	0	0	0
9	1615	${}^{1^{1156}2^{341}3^54^{55}6^{15}8^{19}9^112^{10}}_{16^418^124^232^236^148^1}_{108^1288^1}$	1081	492	41	1	0	0	0
10	25,180	$\begin{array}{c} 1^{21970}2^{2533}3^{24}4^{421}5^{1}6^{44}8^{102} \\ 10^{2}12^{16}16^{25}18^{1}20^{4}24^{11} \\ 32^{9}36^{2}48^{4}64^{1}96^{1}120^{2} \\ 128^{1}288^{4}576^{1}1728^{1} \end{array}$	17,038	7426	688	26	2	0	0
11	479,238	$1^{454542}2^{21449}3^{41}4^{2494}6^{67}8^{456}\\11^{1}12^{40}16^{72}24^{27}32^{14}\\36^{1}48^{12}64^{2}72^{2}96^{6}\\144^{7}288^{3}864^{1}4032^{1}$	323,591	142,075	13,193	371	8	0	0
12	10,695,820	$1^{10431210}2^{236254}3^{229}4^{23407}\\ 6^{267}8^{3234}12^{224}16^{510}18^{3}\\ 24^{164}32^{100}36^{7}48^{97}64^{20}\\ 72^{11}96^{30}128^{7}144^{9}\\ 192^{8}288^{8}384^{7}432^{1}\\ 576^{1}768^{2}864^{2}1152^{3}\\ 1536^{1}1728^{1}2016^{1}\\ 10368^{1}82944^{1}\\ 10368^{1}8294^{1}\\ 10868^{$	7,087,335	3,289,199	308,659	10,447	170	9	1
13	270,939,475	$1^{267284483}2^{3338204}3^{1034}4^{285033}\\ 6^{1388}8^{22125}12^{1748}13^1\\ 16^{2502}18^924^{1713}32^{403}36^{15}\\ 39^{1}48^{501}64^{68}72^{28}\\ 96^{90}128^{10}144^{35}192^{30}\\ 216^{1}256^{3}288^{16}336^{1}\\ 384^{10}432^{6}576^{4}768^{3}\\ 864^{3}1152^{1}2016^{2}2592^{1}\\ 4032^{1}12096^{1}13824^{1}\\ \end{cases}$	175,420,488	87,098,667	810,0133	315,860	4266	60	1

 TABLE 1
 Sizes of generating sets of full *n*-line configurations

In the current study, this approach is used to extend the earlier results to n = 8. Notice that the following theorem actually confirms the old results for $n \le 7$, including the unpublished result [27] for n = 7.

Theorem 7. For $n \le 8$, there is no full n-line configuration C whose number of occurrences in an STS(v) is a polynomial in v plus a linear combination (with coefficients that are polynomials in v) of the numbers of occurrences of the full m-line configurations with $m \le n$ excluding C.

Proof. For n = 8, the 623 distinct STS(25)s listed in [17] were considered. As there are 179 full *m*-line configurations with $m \le 8$, vectors of length 180 were determined as described earlier; these vectors are also listed in [17]. Calculations using gap now show that the 623 × 180 matrix formed by these vectors has rank 180.

See [1,6,14] for further results on configurations in designs in general and in Steiner triple systems in particular.

2.3 $\mid w_3$ Configurations

For configurations with a large number of lines, a more general study is feasible only for specific types of configurations. We here consider w_3 configurations. The number N of isomorphism classes of w_3 configurations with small w can be found in [2,16]. In Table 2 we list those values for $7 \le n \le 16$, and for each entry we give the same information as in Table 1.

The unique 7_3 and 8_3 configurations are the Fano plane and the Möbius–Kantor configuration, respectively, and are depicted in Figure 3, again with generating sets indicated.

We call an n_3 configuration in a v_3 configuration a *subconfiguration* and say that such a subconfiguration is *proper* if n < v.

Theorem 8. If a v_3 configuration has a proper n_3 subconfiguration, then it has a proper $(n - v)_3$ subconfiguration.

Proof. Consider the configuration obtained by removing the points and lines of a proper n_3 subconfiguration from the v_3 configuration.

w	Ν	Aut	N_3	N_4	N_5	N_6
7	1	168 ¹	1	0	0	0
8	1	48 ¹	1	0	0	0
9	3	9 ¹ 12 ¹ 108 ¹	3	0	0	0
10	10	$2^1 3^2 4^2 6^1 10^1 12^1 24^1 120^1$	9	1	0	0
11	31	$1^{10}2^{13}3^{1}4^{2}6^{3}8^{1}11^{1}$	31	0	0	0
12	229	$1^{\!146}2^{60}3^34^36^88^112^318^124^132^136^172^1$	224	5	0	0
13	2036	$1^{1770} 2^{190} 3^{20} 4^{30} 6^{16} 8^3 12^4 13^1 39^1 96^1$	2010	26	0	0
14	21,399	${}^{20328}2^{916}3^{19}4^{91}6^{12}7^{1}8^{15}12^{7}14^{3}16^{3}24^{2}\\128^{1}56448^{1}$	20,798	599	1	1
15	245,342	$\frac{1^{241240}2^{3709}3^{69}4^{180}5^56^{59}8^{34}10^312^{11}15^216^{10}}{18^120^224^230^232^{1}48^672^{1}128^{1}192^2}{720^18064^1}$	222,524	22,809	8	1
16	3,004,881	$\frac{1^{2986560}2^{17119}3^{320}4^{635}6^{88}8^{93}12^{19}16^{24}}{24^{7}32^{5}48^{5}96^{2}1512^{1}2016^{1}4608^{1}18144^{1}}$	2,260,797	744,045	35	4

TABLE 2	Sizes of	generating	sets of w_3	configurations
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FIGURE 3 The 7₃ and 8₃ configurations

Corollary 1. If a v_3 configuration has a proper n_3 subconfiguration, then $v \ge 14$.

The smallest example of a v_3 configuration with proper n_3 subconfigurations is of type 14₃ and is unique as the 7₃ configuration is unique. This particular configuration occurs in the STS(21)s of Wilson type, as discussed in [18].

The size of the generating set of a v_3 configuration with proper n_3 and $(v - n)_3$ subconfigurations equals the sum of the sizes of the generating sets of those subconfigurations. All configurations corresponding to the entries in column N_6 of Table 2 can be explained in this way. This argument can be applied recursively.

Theorem 9. For any integer d, there is a w_3 configuration whose smallest generating set has size greater than d.

Proof. Consider the $(7m)_3$ configuration consisting of m 7₃ subconfigurations. Each of the 7₃ subconfigurations requires three points for generation, so the minimum number of points in a generating set is 3m.

3 | **PRACTICAL ALGORITHMS**

There are two situations when fast practical algorithms for counting configurations in Steiner triple systems are needed: if there are many Steiner triple systems to consider, as in [8], or if the order of the Steiner triple systems is large.

The main challenge in this study is that—as we are interested on average-case performance—in a formal analysis one should know the distribution of all possible inputs. An experimental approach was taken here, and algorithms were evaluated using random Steiner triple systems. The algorithms were produced in an exhaustive manner that will be described later in this section. This falls within the paradigm of using algorithms to design algorithms [4]. Hopefully, the computational results will also inspire analytical studies of these algorithms.

In this section, we apply the following conventions. For a Steiner triple system (V, \mathcal{B}) , we let $V = \{0, 1, ..., v - 1\}$. In (V, \mathcal{B}) , we want to count the number of occurrences of a configuration (V', \mathcal{B}') with |V'| = w points, $|\mathcal{B}'| = b$ lines, and a minimum generating set of size *m*.

We use two auxiliary functions $B_2 : \{(x, y) \in V^2 : x \neq y\} \to V$ and $B_3 : V^3 \to \{0, 1\}$ so that $B_2(x, y) = z$ iff $\{x, y, z\} \in \mathcal{B}$ and

$$B_3(x, y, z) = \begin{cases} 0, & \text{if } \{x, y, z\} \notin \mathcal{B}, \\ 1, & \text{if } \{x, y, z\} \in \mathcal{B}. \end{cases}$$

A precomputed array of size v^2 can be used to evaluate both of these functions in constant time.

The configurations (V', \mathcal{B}') we focus on are the w_3 configurations with $w \le 8$ and the full *n*-line configurations with $n \le 6$, that is, the nine configurations depicted in this paper.

3.1 Generating sets, up to symmetry

The main idea in the algorithms to be considered is that they loop over values for elements in a generating set of size *m*. A configuration may have many such generating sets, but the number of generating sets to consider can be reduced by utilizing symmetries of the configuration, that is, its automorphism group Γ . Since the nesting of the loops in the algorithms implies an order on the elements of a generating set, we consider precisely the nestings given by representatives from the set \mathcal{M} of transversals of the action of Γ on the *ordered* generating sets.

The automorphism group of a configuration can also be utilized to derive conditions on its points, so that occurrences will not be counted multiple times. We denote the orbit of an element *x* under the action of *G* by $G \cdot x$ and the stabilizer by $\operatorname{Stab}_G(x)$. Let $Z = (z_1, z_2, ..., z_m)$ be a permutation of an element $M \in \mathcal{M}$. For $i \in \{1, 2, ..., m\}$, we now compute $O_i = \Gamma_{i-1} \cdot z_i$ and $\Gamma_i = \operatorname{Stab}_{\Gamma_{i-1}}(z_i)$, using $\Gamma_0 = \Gamma$. Whereas *M* gives the nesting of the **for** loops, z_i and O_i show where the conditions given by the symmetries will be taken into account. Indeed, we require z_i to be the smallest or largest element in the orbit O_i . The value $Q := |\Gamma_m|$ gives the number of times each configuration will be encountered.

To get an exhaustive set of algorithms, for each $M \in \mathcal{M}$, we consider each of the *m*! possible choices of *Z* as permutations of *M* and each of the 2^m possible choices of $E = (e_1, e_2, ..., e_m)$, $e_i \in \{\min, \max\}$, where e_i tells whether the corresponding z_i should be minimum or maximum in the orbit O_i . As the isomorphism $i \mapsto v - 1 - i$ does not change the average-case behavior, e_1 can be fixed to min, which leaves 2^{m-1} possible choices.

The following concrete example, which is split into two parts, demonstrates how one algorithm for the Fano plane is obtained.

Example (Fano plane, Figure 3A). The lines \mathcal{B}' of the Fano plane in Figure 3A are {{a, b, e}, {a, c, g}, {a, d, f}, {b, c, f}, {b, d, g}, {c, d, e}, {e, f, g}}. Given a Steiner triple system, the goal is now to set the seven points a, b, c, d, e, f, g, which we regard as variables, such that ({a, b, c, d, e, f, g}, \mathcal{B}') is a configuration of the system.

The Fano plane has w = 7 points, b = 7 lines, an automorphism group Γ of order $|\Gamma| = 168$, and a minimum generating set of size m = 3. It has 28 minimum generating sets, which are precisely the sets of three noncollinear points, so there are $28 \cdot 3! = 168$ sets of ordered minimum generating sets. Those 168 sets form one orbit under the action of Γ , and we can let $\mathcal{M} = \{(a, b, c)\}$. Hence, exactly one way of nesting the **for** loops is considered: a - b - c.

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One possible choice for Z and E is Z = (c, b, a) and $E = (\min, \min, \min)$. We then get $O_1 = \{a, b, c, d, e, f, g\}$ (the automorphism group of the Fano plane is pointtransitive), $|\Gamma_1| = 24$, $O_2 = \{a, b, d, e, f, g\}$, $|\Gamma_2| = 4$, $O_3 = \{a, d, e, g\}$, and $|\Gamma_3| = Q = 1$. The constraints are $c = \min\{a, b, c, d, e, f, g\}$, $b = \min\{a, b, d, e, f, g\}$, and $a = \min\{a, d, e, g\}$, which can be simplified to c < b < a < d, e, g and b < f.

Note that if $|\Gamma_i| = 1$ for some i < m, then $|\Gamma_j| = 1$ and $|O_j| = 1$ for j > i. The minimum and maximum of a 1-element set coincide, and we then get identical algorithms regardless of the values of e_j and z_j for j > i. Obviously, there are then no additional constraints on some **for** loop variables. This situation occurs especially for configurations with very small automorphism groups, such as the crown configuration (which has automorphism group order 2). For comparison, we actually also included in our experiments all variants of algorithms where, for $i \in \{1, 2, ..., m\}$, Γ_j and O_j are not determined for $j \ge i$. Then $Q := |\Gamma_{i-1}|$ and some **for** loop variables do not have constraints. These variants are included later in the last column of Table 3 but are not further discussed here as they were not successful in the experimental evaluation.

3.2 | Algorithm for constructing counting algorithms

We are now ready to discuss our approach for exhaustively constructing algorithms for counting (V', \mathcal{B}') configurations in (V, \mathcal{B}) . For clarity, we only consider the case of generating sets of size 3, but the approach can be extended to arbitrary sizes of generating sets with further nesting of **for** loops (**continue** forces the next iteration of the **for** loop to take place). The missing part, Ω , is explained later.

$r \leftarrow 0$	// A_1
for $x_{\mathbf{F}_1} \leftarrow y_{\mathbf{F}_1}$ to $z_{\mathbf{F}_1}$ do	// F $_1$
for $x_{\mathbf{F}_2} \leftarrow y_{\mathbf{F}_2}$ to $z_{\mathbf{F}_2}$ do	// F $_2$
if $\mathbf{Check}(F_2) = 0$ continue	// C_{F_2}
$x_{\mathbf{S}_1} \leftarrow B_2(x_{\mathbf{F}_1}, x_{\mathbf{F}_2})$	// S $_1$
if $\mathbf{Check}(S_1) = 0$ continue	// C _{S1}
for $x_{\mathbf{F}_3} \leftarrow y_{\mathbf{F}_3}$ to $z_{\mathbf{F}_3}$ do	// F ₃
if $\mathbf{Check}(F_3) = 0$ continue	// C _{F3}
Ω	
$r \leftarrow r+1$	// A_2
return r/Q	// A ₃

The comments in the right margin of the algorithm describe the type of action taken in the respective place.

- A_i Actions on the accumulator *r* for counting configurations (*i* = 1: initializing; *i* = 2: increasing; *i* = 3: final division as each configuration is seen *Q* times)
 C_i Checks after setting variable x_i
- D_B Checks regarding existence of line *B*

(Continues)

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 F_i for loop for variable x_{F_i}

 S_i Fixing variable x_{S_i}

The variables x_i contain the points of the configurations; these are either inside (x_{F_i} , *m* variables) or outside (x_{S_i} , w - m variables) a generating set. Let us next elaborate on the main details.

- 1. In the **for** loops, the values of y_{F_i} and z_{F_i} are set based on constraints on elements being minimum or maximum in orbits, as discussed in Section 3.1. With no restrictions, we would have $y_{F_i} = 0$ and $z_{F_i} = v 1$; these can be somewhat increased and decreased, respectively, when we have a lower bound on the number of elements that are smaller or larger, respectively.
- 2. In the test Check(X), we incorporate further tests of elements being minimum or maximum in orbits. Some of the tests are included in the for loops, as discussed in Item 1; any other inequalities that can be tested are included here. Whenever a point outside the generating set is fixed, all such tests are carried out here. Moreover, we need to make sure that a point that is fixed differs from the points that have been fixed earlier. The situation that a point is not new may occur both in a for loop and when fixing a point in S_i.
- 3. Whenever a variable is fixed, we check the existence of the lines *B* of the configuration that have not been involved so far in determining new points or lines D_B but whose points are fixed. An early test makes sense since the probability of existence of a particular line in a random STS is 1/(v 2).

The missing part in the algorithm, Ω , is now as follows. For each variable x_{S_i} that has not yet been set, we have a code line of type S_i to assign a value to x_{S_i} . Then we have a code line of type C_{S_i} to test whether all constraints are fulfilled and whether x_{S_i} differs from all points that have been fixed earlier. Finally, we have one code line of type D_B for each line in the configuration that consists of fixed points but did not occur in a code line of type S_i or D_B so far.

Note that there may be several ways of building up a configuration from a generating set, and we consider all possible such ways in the construction of algorithms.

We can now finish the example for Fano planes that we started in Section 3.1.

Example (cont.). On the basis of the calculations earlier in the example, in particular the constraints c < b < a < d, e, g and b < f, we get the following overall structure of the algorithm, where the parts Ω_1 and Ω_2 are yet to be determined:

```
r \leftarrow 0<br/>for a \leftarrow 2 to v - 4 do<br/>for b \leftarrow 1 to a - 1 do<br/> \begin{bmatrix} \Omega_1 \\ \text{for } c \leftarrow 0 \text{ to } b - 1 \text{ do} \\ \Omega_2 \\ r \leftarrow r + 1 \end{bmatrix}
```

```
return r
```

Next, we will discuss the remaining w - m = 4 variables d, e, f, g. The sets of new points in Ω_1 and Ω_2 are $G_1 = \{e\}$ and $G_2 = \{d, f, g\}$, respectively. We here consider one feasible ordering of the elements of G_1 and G_2 : (e) and (g, d, f), respectively.

In Ω_1 , there is only one way of setting e, namely, {a, b, e}, and the only constraint is a < e. This also ensures distinctness of all points so far, so for Ω_1 we get

 $e \leftarrow B_2(b, a)$ if $e \le a$ continue

In Ω_2 , one choice for setting g, d, and f is {a, c, g}, {c, d, e}, and {e, f, g}, respectively. This choice implies that the remaining lines to check are {a, d, f}, {b, c, f}, and {b, d, g}.

For the loop variable c we need to ensure that c < b. The tests related to g, d, and f are a < g, a < d, and b < f, respectively. These are also sufficient, as it can be verified that all points obtained in this way are necessarily distinct (e.g., $g \neq e$ as {a, c, g} and {a, b, e} are distinct lines through a and $d \neq e$ as {c, d, e} is a line). Altogether, for Ω_2 we have

```
g \leftarrow B_2(c, a)

if g \leq a continue

d \leftarrow B_2(e, c)

if d \leq a continue

if B_3(b, d, g) = 0 continue

f \leftarrow B_2(g, e)

if f \leq b continue

if B_3(a, d, f) = 0 continue

if B_3(b, c, f) = 0 continue
```

We then get the following complete algorithm.

```
r \leftarrow 0
for a \leftarrow 2 to v - 4 do
    for b \leftarrow 1 to a - 1 do
         e \leftarrow B_2(b, a)
         if e < a continue
         for c \leftarrow 0 to b - 1 do
            g \leftarrow B_2(c, a)
            if g \leq a continue
            d \leftarrow B_2(e, c)
            if d \leq a continue
            if B_3(b, d, g) = 0 continue
            f \leftarrow B_2(g, e)
            if f \leq b continue
             if B_3(a, d, f) = 0 continue
             if B_3(b, c, f) = 0 continue
             r \leftarrow r + 1
```

return r

Algorithm 1: 7₃-configuration (Fano plane) (Figure 3A)

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3.3 | Experimental evaluation

We constructed all possible algorithms with the approach discussed earlier and carried out an experimental evaluation. The authors are well aware of the challenges involved in such work. Above all, the performance of the algorithms depends on issues that are difficult or impossible to control, related to compilers and microprocessors. In particular, the programs contain many **if** statements and can be demanding for the technique of speculative execution used commonly by modern central processing units [21].

Table 3 summarizes the main details for the generation process of algorithms for all configurations in consideration here. The column "Name" is the name of the configuration, and b, w, m, $|\Gamma|$, $|\mathcal{M}|$, and $|\mathcal{M}/\Gamma|$ are the number of lines, points, elements in a generating set of minimum size, automorphisms, minimum ordered generating sets, and orbits of minimum ordered generating sets under the action of the automorphism group Γ , respectively. The last column A gives the number of distinct algorithms generated.

For each configuration, each of the A algorithms was evaluated on random Steiner triple systems of orders 93, 121, and 151 constructed by Stinson's hill-climbing algorithm [24]. Due to the large number of algorithms to be considered, the evaluation took place in several phases, gradually decreasing the number of algorithms and increasing the amount of time used for each algorithm.

We conclude the paper by listing the nine algorithms obtained in the aforementioned manner as Algorithms 1 to 9; Algorithm 1 is given in the example in Section 3.2. Any scholar needing such algorithms should be able to implement them easily in any programming language.

Name	b	w	т	$ \Gamma $	$ \mathcal{M} $	$ \mathcal{M}/\Gamma $	A
Pasch	4	6	3	24	96	4	296
Mitre	5	7	3	12	180	15	1272
Fano-line	6	7	3	24	168	7	2020
Crown	6	8	3	2	276	138	7348
Hexagon	6	8	3	12	288	24	2912
Prism	6	9	4	12	1800	150	60,872
Grid	6	9	4	72	1944	27	34,752
Fano	7	7	3	168	168	1	828
Möbius–Kantor	8	8	3	48	288	6	9216

TABLE 3 Details for the generated algorithms

 $\begin{array}{c|c} r \leftarrow 0 \\ \textbf{for } \mathbf{a} \leftarrow 0 \ \textbf{to} \ v - 6 \ \textbf{do} \\ \hline \textbf{for } \mathbf{b} \leftarrow \mathbf{a} + 1 \ \textbf{to} \ v - 2 \ \textbf{do} \\ \mathbf{e} \leftarrow B_2(\mathbf{b}, \mathbf{a}) \\ \textbf{if } \mathbf{e} \leq \mathbf{a} \ \textbf{continue} \\ \textbf{for } \mathbf{f} \leftarrow \max\{\mathbf{b}, \mathbf{e}\} + 1 \ \textbf{to} \ v - 1 \ \textbf{do} \\ \hline \mathbf{c} \leftarrow B_2(\mathbf{f}, \mathbf{a}) \\ \textbf{if } \mathbf{f} \leq \mathbf{c} \lor \mathbf{c} \leq \mathbf{a} \ \textbf{continue} \\ \textbf{d} \leftarrow B_2(\mathbf{f}, \mathbf{e}) \\ \textbf{if } \mathbf{d} \leq \mathbf{a} \ \textbf{continue} \\ \textbf{if } B_3(\mathbf{b}, \mathbf{c}, \mathbf{d}) = 0 \ \textbf{continue} \\ r \leftarrow r + 1 \end{array}$

return r

Algorithm 2: Pasch configuration (Figure 1A)

$$\begin{array}{c|c} r \leftarrow 0 \\ \text{for } \mathbf{a} \leftarrow 0 \text{ to } v - 3 \text{ do} \\ \hline \\ \text{for } \mathbf{c} \leftarrow \mathbf{a} + 1 \text{ to } v - 2 \text{ do} \\ \text{for } \mathbf{c} \leftarrow \mathbf{a} + 1 \text{ to } v - 2 \text{ do} \\ \text{for } \mathbf{e} \leftarrow B_2(\mathbf{c}, \mathbf{a}) \\ \text{for } \mathbf{e} \leftarrow \max\{\mathbf{c}, \mathbf{f}\} + 1 \text{ to } v - 1 \text{ do} \\ \hline \\ & g \leftarrow B_2(\mathbf{f}, \mathbf{e}) \\ \text{b} \leftarrow B_2(\mathbf{g}, \mathbf{a}) \\ \text{if } \mathbf{e} \leq \mathbf{b} \text{ continue} \\ \text{d} \leftarrow B_2(\mathbf{g}, \mathbf{c}) \\ \text{if } \mathbf{e} \leq \mathbf{d} \text{ continue} \\ \text{if } B_3(\mathbf{b}, \mathbf{d}, \mathbf{e}) = 0 \text{ continue} \\ r \leftarrow r + 1 \end{array}$$

return r

Algorithm 3: Mitre configuration (Figure 1B)

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 $\begin{array}{c|c} r \leftarrow 0 \\ \textbf{for } \textbf{c} \leftarrow 2 \ \textbf{to} \ v - 2 \ \textbf{do} \\ \hline \textbf{for } \textbf{e} \leftarrow 0 \ \textbf{to} \ \textbf{c} - 2 \ \textbf{do} \\ \hline \textbf{b} \leftarrow B_2(\textbf{c}, \textbf{e}) \\ \textbf{for } \textbf{f} \leftarrow \textbf{e} + 1 \ \textbf{to} \ \textbf{c} - 1 \ \textbf{do} \\ \hline \textbf{if } \textbf{f} = \textbf{b} \ \textbf{continue} \\ \textbf{g} \leftarrow B_2(\textbf{f}, \textbf{b}) \\ \textbf{if } \textbf{g} \leq \textbf{c} \ \textbf{continue} \\ \textbf{d} \leftarrow B_2(\textbf{e}, \textbf{g}) \\ \textbf{if } B_3(\textbf{c}, \textbf{d}, \textbf{f}) = 0 \ \textbf{continue} \\ \textbf{a} \leftarrow B_2(\textbf{f}, \textbf{e}) \\ \textbf{if } B_3(\textbf{a}, \textbf{c}, \textbf{g}) = 0 \ \textbf{continue} \\ \textbf{r} \leftarrow r + 1 \end{array}$

return r

Algorithm 4: Fano-line configuration (Figure 2A)

$$\begin{array}{c|c} r \leftarrow 0 \\ \textbf{for } \mathbf{f} \leftarrow 0 \ \textbf{to} \ v - 2 \ \textbf{do} \\ \hline \mathbf{for } \mathbf{f} \leftarrow 0 \ \textbf{to} \ v - 1 \ \textbf{do} \\ \hline \mathbf{for } \mathbf{e} \leftarrow 0 \ \textbf{to} \ v - 1 \ \textbf{do} \\ \hline \mathbf{h} \leftarrow B_2(\mathbf{f}, \mathbf{e}) \\ \mathbf{for } \mathbf{g} \leftarrow \mathbf{f} + 1 \ \textbf{to} \ v - 1 \ \textbf{do} \\ \hline \mathbf{h} \ \mathbf{f} \ \mathbf{g} \in \{\mathbf{e}, \mathbf{h}\} \ \textbf{continue} \\ \mathbf{d} \leftarrow B_2(\mathbf{e}, \mathbf{g}) \\ \mathbf{b} \leftarrow B_2(\mathbf{f}, \mathbf{d}) \\ \mathbf{c} \leftarrow B_2(\mathbf{h}, \mathbf{g}) \\ \mathbf{if } \ \mathbf{c} = \mathbf{b} \ \textbf{continue} \\ \mathbf{a} \leftarrow B_2(\mathbf{f}, \mathbf{g}) \\ \mathbf{if } B_3(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0 \ \textbf{continue} \\ r \leftarrow r + 1 \end{array}$$

return r

Algorithm 5: Crown configuration (Figure 2E)

```
\begin{array}{c|c} r \leftarrow 0 \\ \text{for } \mathbf{b} \leftarrow 0 \text{ to } v - 6 \text{ do} \\ \hline \\ \text{for } \mathbf{c} \leftarrow \mathbf{b} + 2 \text{ to } v - 1 \text{ do} \\ \mathbf{a} \leftarrow B_2(\mathbf{c}, \mathbf{b}) \\ \text{for } \mathbf{d} \leftarrow \mathbf{b} + 1 \text{ to } \mathbf{c} - 1 \text{ do} \\ \hline \\ \text{if } \mathbf{d} = \mathbf{a} \text{ continue} \\ \mathbf{e} \leftarrow B_2(\mathbf{d}, \mathbf{a}) \\ \text{if } \mathbf{e} \leq \mathbf{b} \text{ continue} \\ \mathbf{h} \leftarrow B_2(\mathbf{d}, \mathbf{b}) \\ \mathbf{g} \leftarrow B_2(\mathbf{h}, \mathbf{c}) \\ \text{if } \mathbf{g} \leq \mathbf{b} \lor \mathbf{g} = \mathbf{e} \text{ continue} \\ \mathbf{f} \leftarrow B_2(\mathbf{h}, \mathbf{e}) \\ \text{if } \mathbf{f} \leq \mathbf{b} \text{ continue} \\ \text{if } B_3(\mathbf{a}, \mathbf{f}, \mathbf{g}) = 0 \text{ continue} \\ r \leftarrow r + 1 \end{array}
```



Algorithm 6: Hexagon configuration (Figure 2D)

```
r \leftarrow 0
for a \leftarrow 1 to v - 2 do
    for f \leftarrow 0 to a - 1 do
         for b \leftarrow 0 to v - 1 do
             if b \in \{a, f\} continue
             e \leftarrow B_2(b, a)
             if e = f continue
             c \leftarrow B_2(f, b)
             h \leftarrow B_2(e, c)
             if h \leq a continue
             for d \leftarrow e + 1 to v - 1 do
                  if d \in \{a, b, c, f, h\} continue
                  g \leftarrow B_2(d, a)
                  if g \in \{c, f, h\} continue
                  i \leftarrow B_2(h,d)
                  if i \in \{b, f\} continue
                  if B_3(f, g, i) = 0 continue
                  r \leftarrow r + 1
```

return *r* **Algorithm 7:** Prism configuration (Figure 2C)

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```
r \leftarrow 0
for a \leftarrow 0 to v - 9 do
     for b \leftarrow a + 1 to v - 3 do
          d \leftarrow B_2(b, a)
          if d \leq b continue
          for e \leftarrow d + 1 to v - 1 do
               g \leftarrow B_2(e, a)
               \mathbf{if} \ \mathbf{e} \leq \mathbf{g} \lor \mathbf{g} \leq \mathbf{a} \ \mathbf{continue}
               for c \leftarrow a + 1 to v - 1 do
                     if c \in \{b, d, e, g\} continue
                     f \leftarrow B_2(c, b)
                     if f \leq a \lor f \in \{e, g\} continue
                     h \leftarrow B_2(e, c)
                    if h \leq a \lor h = d continue
                     i \leftarrow B_2(g, f)
                     if i \leq a \lor i \in \{d, h\} continue
                    if B_3(d, h, i) = 0 continue
                     r \leftarrow r+1
```

```
return r
```

Algorithm 8: Grid configuration (Figure 2B)

```
r \leftarrow 0
for a \leftarrow 1 to v - 6 do
    for b \leftarrow a + 1 to v - 1 do
         g \leftarrow B_2(b, a)
         if g \leq a continue
         for c \leftarrow 0 to a - 1 do
             d \leftarrow B_2(c, a)
             if d \leq a continue
             h \leftarrow B_2(c, b)
             if h \leq a continue
             e \leftarrow B_2(d, b)
             if e \leq c continue
             if B_3(e, g, h) = 0 continue
             f \leftarrow B_2(e, a)
             if f \leq a continue
             if B_3(c, f, g) = 0 continue
             if B_3(d, f, h) = 0 continue
             r \leftarrow r+1
```



Algorithm 9: 8₃-configuration (Möbius–Kantor) (Figure 3B)

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