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Published in:
IEEE Communications Letters

DOI:
10.1109/LCOMM.2016.2615300

Published: 01/01/2017

Document Version
Peer reviewed version

Packet Error Rate Analysis for Uncoded Schemes in Block-Fading Channels using Extreme Value Theory

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***Abstract*—We present a generic approximation of the packet error rate (PER) function of uncoded schemes in the AWGN channel using extreme value theory (EVT). The PER function can assume both the exponential and the Gaussian Q-function bit error rate (BER) forms. The EVT approach leads us to a best closed-form approximation, in terms of accuracy and computational efficiency, of the average PER in block-fading channels. The numerical analysis shows that the approximation holds tight for any value of SNR and packet length whereas the earlier studies approximate the average PER only at asymptotic SNRs and packet lengths.**

***Index Terms*—Packet error rate, block-fading channels, extreme value theory**

I. Introduction

In packet radio systems, the packet error rate (PER) analysis has practical significance for reliability and throughput estimation. Other than noise and packet collisions, therein, modeling the packet errors due to present fading conditions is an important problem. The time scale of fading relative to the bit or packet duration influences the selection of a packet error model. If the wireless channel remains constant across the length of a packet and the consecutive packets observe independent channel realizations, the packet error model must assume constant (block) fading for all the bits in a packet.

Block-fading characterizes the wireless channels that experience slowly varying fading conditions. The PER evaluation in block-fading channels is studied extensively. However, for any modulation and coding scheme, its exact evaluation is complex and usually loose bounding methods as Jenson's inequality and Chernoff upper bound are employed [1].

In [2], the average PER in Rayleigh block-fading, for both uncoded (commonly used in short-range radio systems) and coded schemes, is tightly upper bounded by $1-\exp(-\omega_0/\bar{\gamma})$, where $\bar{\gamma}$ is average signal to noise ratio (SNR) and ω_0 corresponds to inverse coding gain. By definition, $\omega_0 = \int_0^\infty \acute{P}_e(\gamma)d\gamma$, where $\acute{P}_e(\gamma)$ is the PER in the AWGN channel. A log-domain linear approximation of ω_0, for uncoded FSK, is proposed in [1]. In [3], the same model is utilized to estimate the parameters of ω_0 for different modulation schemes. However, the upper bound in [2] and the approximations in [1] [3] are tight only in asymptotic regime: that is, when the average SNR is high or the packet length is large.

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Manuscript received January 20, 2016; accepted September 24, 2016.

In this paper, we present a new analytical method to approximate the PER function in the AWGN channel for uncoded schemes. The method uses the extreme value theory (EVT) to find a limiting distribution of the PER. The limiting distribution is easily integrable over Nakagami-m fading distribution, and yields a best approximation of the average PER in block-fading channels. Our method offers an alternative approach to accurately approximate the average PER of uncoded schemes which either cannot be derived in closed-form or involve computationally extensive calculations.

II. System Model

Let $\mathbf{x}$ be the bits in a packet transmitted over a block-fading channel and $\mathbf{y}$ be the received symbols. Then, the input $\mathbf{x}$ and the output $\mathbf{y}$ of the channel are related as,

$$\mathbf{y} = h\mathbf{x} + \mathbf{n} \tag{1}$$

where h is the instantaneous fading coefficient. It is a zero-mean circularly symmetric complex Gaussian (CSCG) random variable with unit-variance i.e., $\mathbb{E}[|h|^2] = 1$. While, $\mathbf{n}$ is a sequence of mutually independent, zero-mean, CSCG noise with power N_0. If P is power per transmit bit, $\gamma = |h|^2 P/N_0$ is instantaneous received SNR and average SNR is,

$$\bar{\gamma} = \mathbb{E}[\gamma] = \mathbb{E}\left[|h|^2 P/N_0\right] = P/N_0 \tag{2}$$

The average PER over a block-fading channel, $\bar{P}_e(\bar{\gamma})$, is computed by integrating the PER in the AWGN, denoted as $\acute{P}_e(\gamma)$, over the distribution of received SNR, $p(\gamma;\bar{\gamma})$ [2],

$$\bar{P}_e(\bar{\gamma}) = \int_0^\infty \acute{P}_e(\gamma)p(\gamma;\bar{\gamma})d\gamma \tag{3}$$

Depending on the radio propagation environment, the fading magnitude $|h|$ and consequently γ will take different distributions. Whereas, for an N-bit packet, $\acute{P}_e(\gamma)$ is defined as,

$$\acute{P}_e(\gamma) = 1 - \left(1 - b_e(\gamma)\right)^N \tag{4}$$

where $b_e(\gamma)$ is the modulation-dependent instantaneous BER.

We consider $b_e(\gamma)$ with the following generic forms,

$$b_e(\gamma) = c_m Q\left(\sqrt{k_m\gamma}\right) \tag{5}$$

$$b_e(\gamma) = c_m \exp\left(-k_m\gamma\right) \tag{6}$$

where c_m and k_m are the modulation-specific constants. The modulation schemes such as M-ASK, M-PAM, MSK, M-PSK and M-QAM have the BER form of (5) where $Q(\cdot)$ is the Gaussian Q-function. Whereas, FSK and DPSK non-coherent modulations are described by the BER form in (6) [2].

III. EVT based PER Function Approximation

The evaluation of $\bar{P}_e(\bar{\gamma})$ is difficult because of $\acute{P}_e(\gamma)$ expression involving a polynomial of degree N. In what follows, we provide a generic asymptotic approximation of (4) using extreme value theory (EVT) for the BER expressions described in (5) and (6) that allows to evaluate $\bar{P}_e(\bar{\gamma})$ easily.

Proposition 1: The PER in the AWGN channel for the BER functions in (5) and (6), as $N \to \infty$, is approximated by the Gumbel distribution function for the minimum, i.e.,

$$\acute{P}_e(\gamma) \approx 1 - \exp\left(-\exp\left(-\frac{\gamma - a_N}{b_N}\right)\right) \tag{7}$$

where a_N and $b_N > 0$ are the norming constants.

Proof: For a packet of length N, the PER in (4) for both (5) and (6) can be written as,

$$\acute{P}_e(\gamma) = 1 - \Big(1 + c_m\left(F_X(x) - 1\right)\Big)^N \tag{8}$$

where $F_X(x)$ is the cumulative distribution function (CDF) of the standard normal distribution with $x = \sqrt{k_m \gamma}$ for the BER in (5) while it is the CDF of the exponential distribution with $x = \gamma$ and $\lambda = k_m$ for (6).

For $c_m = 1$ (i.e., BPSK modulation), (8) can be expressed in terms of $[F_X(x)]^N$ which can be approximated, as discussed later, with an extreme value distribution. In order to extend the same approach to any c_m, we manipulate (8) as,

$$\begin{aligned}\acute{P}_e(\gamma) &\approx 1 - \Big(1 + c_m \log F_X(x)\Big)^N \\ &\approx 1 - \Big(e^{c_m \log F_X(x)}\Big)^N \\ &\approx 1 - [F_X(x)]^{Nc_m}\end{aligned} \tag{9}$$

where $\log(\cdot)$ is the natural logarithm. Here, the first approximation results from the inequality $\log y \le y - 1$ for $y > 0$, the second from $e^y \ge 1 + y$ and the last approximation uses the identity $y^a = e^{a \log y}$. The approximation in (9) is quite tight and its accuracy increases as N and $\gamma \to \infty$.

By letting $N' = Nc_m$, we handle $[F_X(x)]^{N'}$ in (9) separately. Note that,

$$[F_X(x)]^{N'} = \prod_{i=1}^{N'} \Pr(X_i \le x) = \Pr\left(\max_{1 \le i \le N'}\{X_i\} \le x\right) \tag{10}$$

implies that $[F_X(x)]^{N'}$ can be approximated by one of the extreme value distributions: Gumbel, Fréchet, Weibull [4]. To assimilate this, assume $X_1, \cdots, X_N$ be the i.i.d. random variables drawn from a common distribution function $F(x)$. Let $M_N = \max_{i=1}^{N} X_i$ denotes the maximum of first N random variables. If there exist constants a_N and $b_N > 0$, and a non-degenerate limit distribution $G(x)$ such that the CDF of normalized M_n converges to $G(x)$,

$$\lim_{N \to \infty} \Pr\left(\frac{M_N - a_N}{b_N} \le x\right) \to G(x) \tag{11}$$

then $F(x)$ is said to be in the domain of attraction of $G(x)$.

In order to find the exact limiting distribution, we recall the sufficient condition, stated in [4] [5], for a distribution function $F(x)$ belonging to the domain of attraction of the Gumbel distribution. Let $\omega(F) = \sup\{x : F(x) < 1\}$, and assume that there is a real number x_1 such that for all $x_1 \le x < \omega(F)$, $f(x) = F'(x)$ and $F''(x)$ exist, and $f(x) \neq 0$. If

$$\lim_{x \to \omega(F)} \frac{d}{dx}\left[\frac{1 - F(x)}{f(x)}\right] = 0 \tag{12}$$

then $F(x)$ converges to the Gumbel distribution as $N \to \infty$. The Gumbel distribution function is $G(x) = \exp(-\exp(-\frac{\gamma - a_N}{b_N}))$ and constants a_N and b_N are,

$$\begin{aligned} a_N &= F^{-1}\Big(1 - \frac{1}{N}\Big) \\ b_N &= F^{-1}\Big(1 - \frac{1}{Ne}\Big) - a_N \end{aligned} \tag{13}$$

where e is the base of the natural logarithm, and $F^{-1}(\cdot)$ denotes the inverse of $F(x)$.

It is straightforward to show that the normal and the exponential distributions satisfy the sufficient condition stated in (12). By replacing $[F_X(x)]^{N'}$ in (9) with the Gumbel distribution function completes the proof. $\square$

The constants a_N and b_N for the normal distribution are determined from (13) by using $N = Nc_m$ and the quantile function of the normal distribution, $F^{-1}(p) = \sqrt{2}\,\mathrm{erf}^{-1}(2p-1)$. As $x = \sqrt{k_m \gamma}$, the transformation for γ gives the constants,

$$\begin{aligned} a_N &= \frac{2}{k_m}\Big(\mathrm{erf}^{-1}\Big(1 - \frac{2}{Nc_m}\Big)\Big)^2 \\ b_N &= \frac{2}{k_m}\Big(\mathrm{erf}^{-1}\Big(1 - \frac{2}{Nc_m e}\Big)\Big)^2 - a_N \end{aligned} \tag{14}$$

Similarly for the exponential case, from (13) and the quantile function of the exponential distribution, $F^{-1}(p; k_m) = -\frac{\log(1-p)}{k_m}$, we have,

$$a_N = \frac{\log(Nc_m)}{k_m}, b_N = \frac{1}{k_m} \tag{15}$$

In next section, we show that the proposed approximation in (7) can easily be evaluated under a general fading distribution.

IV. PER over Block-Fading Channels

We consider Nakagami-m fading model for which the SNR, γ, is gamma distributed with the probability density function,

$$p(\gamma; \bar{\gamma}) = \frac{m^m \gamma^{m-1}}{\bar{\gamma}^m \Gamma(m)} \exp\Big(-\frac{m\gamma}{\bar{\gamma}}\Big), \gamma \ge 0 \tag{16}$$

where $0.5 \le m < \infty$ is the fading parameter and $\Gamma(\cdot)$ is the standard gamma function [6, p.892]. Using (7) and (16) in (3),

$$\bar{P}_e(\bar{\gamma}) \approx 1 - \frac{m^m}{\bar{\gamma}^m \Gamma(m)} \int_0^\infty e^{-e^{-\frac{\gamma - a_N}{b_N}}} \gamma^{m-1} e^{-\frac{m\gamma}{\bar{\gamma}}} d\gamma \tag{17}$$

Let $s = \frac{m}{\bar{\gamma}}$ and $G(\gamma) = e^{-e^{-\frac{\gamma - a_N}{b_N}}}$, then the integral in (17) is recognized as the Laplace transform of $\gamma^{m-1} G(\gamma)$, i.e.,

$$\bar{P}_e(\bar{\gamma}) \approx 1 - \frac{m^m}{\bar{\gamma}^m \Gamma(m)} \int_0^\infty e^{-s\gamma} \gamma^{m-1} G(\gamma) d\gamma. \tag{18}$$

Using the Laplace transform of the PDF of the Gumbel distribution, $\mathcal{L}\{g(\gamma)\} = e^{-a_N s}\Gamma(1 + b_N s)$, and the Laplace transform properties, $\mathcal{L}\{G(\gamma)\} = \frac{\mathcal{L}\{g(\gamma)\}}{s}$ and $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$, (18) is evaluated as,

$$\bar{P}_e(\bar{\gamma}) \approx 1 - \frac{m^m(-1)^{m-1}}{\bar{\gamma}^m \Gamma(m)} \frac{d^{m-1}}{ds^{m-1}} \left(\frac{e^{-sa_N}\Gamma(1+sb_N)}{s} \right) \quad (19)$$

For $m = 1$ (Rayleigh fading), (19) easily reduces to,

$$\bar{P}_e(\bar{\gamma}) \approx 1 - e^{-\frac{a_N}{\bar{\gamma}}}\Gamma\left(1 + \frac{b_N}{\bar{\gamma}}\right). \quad (20)$$

By evaluating the derivative in (19) for $m = 2$ and $m = 3$, we get,

$$\bar{P}_e(\bar{\gamma}) \overset{m=2}{\approx} 1 - e^{-\frac{2a_N}{\bar{\gamma}}}\Gamma\left(1 + \frac{2b_N}{\bar{\gamma}}\right)\left(1 + \frac{2a_N}{\bar{\gamma}} - \frac{2b_N}{\bar{\gamma}} \times \psi\left(1 + \frac{2b_N}{\bar{\gamma}}\right)\right) \quad (21)$$

$$\bar{P}_e(\bar{\gamma}) \overset{m=3}{\approx} 1 - \frac{1}{2}e^{-\frac{3a_N}{\bar{\gamma}}}\Gamma\left(1 + \frac{3b_N}{\bar{\gamma}}\right)\Bigg(2 + \frac{9a_N^2}{\bar{\gamma}^2} + \frac{6a_N}{\bar{\gamma}} + \frac{9b_N^2}{\bar{\gamma}^2}\psi\left(1, 1 + \frac{3b_N}{\bar{\gamma}}\right) - \frac{6b_N}{\bar{\gamma}}\psi\left(1 + \frac{3b_N}{\bar{\gamma}}\right) + \frac{9b_N^2}{\bar{\gamma}^2}\psi\left(1 + \frac{3b_N}{\bar{\gamma}}\right)^2 - \frac{18a_N b_N}{\bar{\gamma}}\psi\left(1 + \frac{3b_N}{\bar{\gamma}}\right)\Bigg) \quad (22)$$

where $\psi(\cdot)$ and $\psi(1, \cdot)$ are the digamma and trigamma functions respectively.

V. Numerical Results and Discussion

We validate the proposed approximation of the average PER for uncoded FSK, BPSK and 16-QAM modulation schemes in block-fading. For this purpose, the approximated average PER in (19) is validated against the numerical evaluation of (3), and the approximation error is compared with earlier studies.

The BER functions for non-coherent FSK and BPSK schemes are $\frac{1}{2}\exp(-\gamma/2)$ and $Q\left(\sqrt{2\gamma}\right)$ respectively. The approximated average PER for these schemes is shown in Fig. 1 for $m = 1$, and depicts perfect matching to the numerical results for small and large packets. For 16-QAM, we use the BER approximation for MQAM $\frac{4}{k}\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3k\gamma}{M-1}}\right)$, where M is the constellation size and $k = \log_2 M$ is the number of bits per symbol [7]. The average PER for 16-QAM using (19) shows the same accuracy over m, N and $\bar{\gamma}$ in Fig. 2.

It is worth noting that one can easily find an exact expression of the average PER for FSK by using the binomial expansion of (4) with the BER in (3). For instance, in Rayleigh fading with $p(\gamma; \bar{\gamma}) = \exp(-\gamma/\bar{\gamma})$, we have,

$$\bar{P}_e(\bar{\gamma}) = 1 - \sum_{n=0}^{N} \binom{N}{n} (-1)^n \frac{(c_m)^n}{1 + nk_m\bar{\gamma}}. \quad (23)$$

However, this expression requires the addition of terms, with alternating sign, each obtained from the multiplication of a very large number with a very small number. Consequently, it is numerically difficult to evaluate (23) for large values of N.

Similarly, the Gaussian Q-function approximations (e.g., [8], [9]) experience the same above-stated difficulty in evaluating the average PER of modulation schemes described by (5). The other approach is to utilize the approximations for an integer power of the Gaussian Q-function (e.g., see [10]). Such approximations are required to evaluate the average symbol error probability (SEP) in fading with SEP in

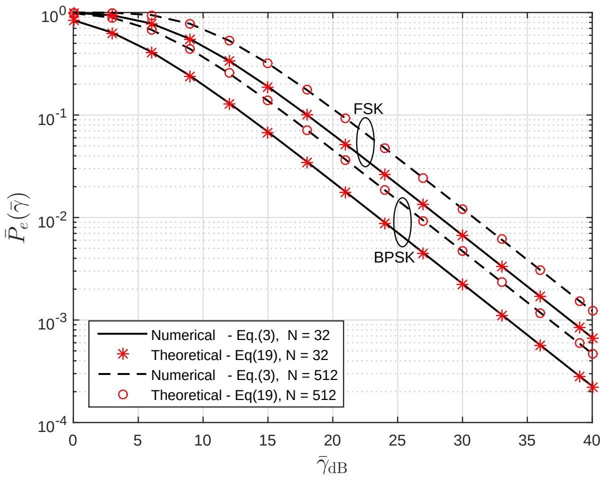

Fig. 1. Average PER for FSK and BPSK modulations in Rayleigh block-fading channel (FSK : $c_m = 1/2, k_m = 1/2$; BPSK : $c_m = 1, k_m = 2$).

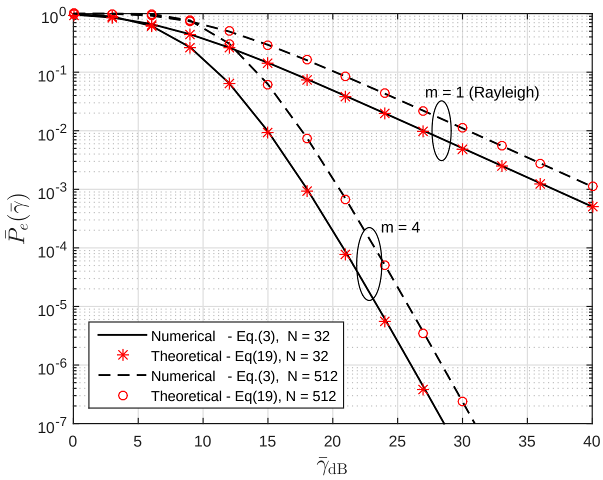

Fig. 2. Average PER for uncoded 16-QAM in Nakagami-m block-fading channel ($c_m = 3/4, k_m = 4/5$).

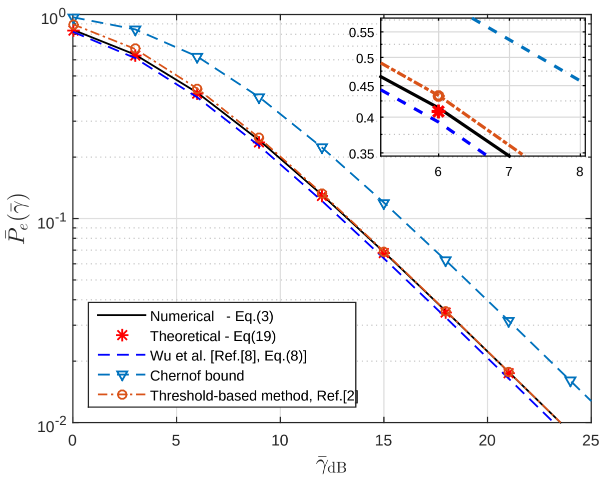

Fig. 3. A comparison of different approximation techniques: average PER of BPSK modulation in Rayleigh block-fading with $N = 32$.

the form of $Q^N(x)$: e.g., differential encoded QPSK where maximum $N = 4$. The approximation developed in [10] is integrable in Nakagami-m fading for any m. However, it requires summation over all sequences of nonnegative integers $k_1, \cdots, k_{n_a}$ such that $k_1 + \cdots + k_8 = N$, which is computationally demanding even for small values of N. In comparison, (19) is easy to compute for any N.

We analyze the average PER for BPSK in Rayleigh fading under the exponential function based approximations of the Gaussian Q-function (Chernof bound and Wu et al. lower bound [8]), threshold-based bound [2] and the proposed approximation in Fig. 3 for $N = 32$. Figure 3 shows that the proposed approximation and threshold-based method match the numerical results tightly, and the threshold-based method is a good candidate for further comparison.

For some specific modulations, closed-form approximations of w_0 in the upper bound on average PER in Rayleigh fading, i.e., $\bar{P}_e(\bar{\gamma}) \leq 1 - \exp(-\omega_0/\bar{\gamma})$ [2], are proposed in [1] and [3]. In [1], w_0 for the BER of form (6) is approximated as,

$$\omega_0 \approx k_1 \log N + k_2 \tag{24}$$

where $k_1 = 1/k_m$ and $k_2 = (\gamma_e + \log(c_m))/k_m$, and $\gamma_e = 0.5772$ is the Euler constant. In [1], for PER evaluation of BPSK, the Gaussian Q-function is approximated with an exponential function and parameters k_1 and k_2 are calculated as above. In [3], ω_0 is estimated by fitting the linear model in (24) to the simulations and, k_1 and k_2 are determined for different modulation schemes. For uncoded 16-QAM, $k_1 = 2.327$ and $k_2 = -3.736$ (see [3], Table I).

In Fig. 4, the approximation error in [1], [3] and (19) for BPSK and 16-QAM in Rayleigh fading is compared. It can be observed that (19) is quite accurate across any value of SNR and packet length. The error for 16-QAM is slightly higher than BPSK, because of the approximations in (9), however this gap reduces quickly with increase in N. Figure 4 also shows that (19) is nearly insensitive to SNR and its accuracy improves mainly with increase in packet length. On the other hand, both [1] and [3] yield large errors at small packet lengths even at high SNRs. Although, the errors in these studies decrease with increase in packet length but still remain higher than the proposed method. The approximation (19) has the same accuracy for FSK (not shown here), and the approximation error at $N = 256$ is less than 0.04%.

In order to eliminate the effect of inaccuracies introduced by approximations of w_0 in [1] and [3], we evaluate w_0 numerically and find the average PER from [2]. For 16-QAM in Rayleigh fading, the resulting error is compared with the one in (19) in Fig. 5. It can be seen that the upper bound in [2] is accurate for large values of SNR and packet length while (19) has much better accuracy at low SNR for any packet lengths. Even at high SNRs, the error in (19) is acceptably small and it reduces sharply with increase in packet length.

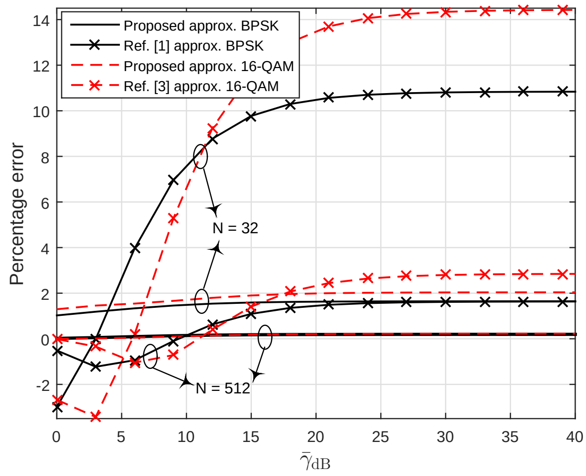

Fig. 4. Approximation error in average PER for uncoded BPSK and 16-QAM schemes in Rayleigh block-fading channel.

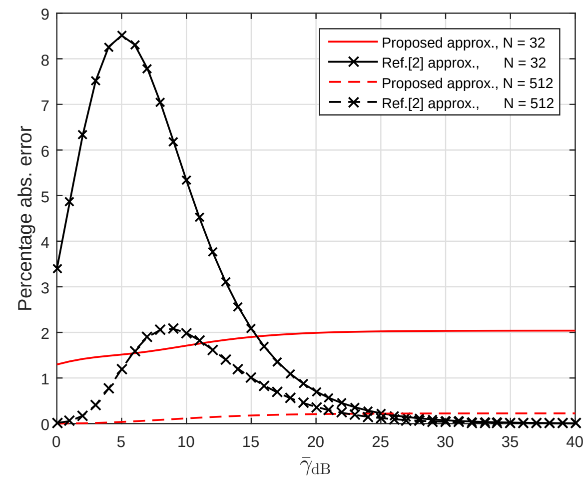

Fig. 5. Approximation error in average PER for uncoded 16-QAM in Rayleigh block-fading: comparing approximation (19) with upper bound on PER [2].

VI. Conclusions

We showed that the PER of uncoded schemes, involving the BER forms of the exponential and the Gaussian Q-function, in the AWGN channel can be asymptotically approximated using EVT. Interestingly, the asymptotic distribution of the PER equals the Gumbel distribution function for sample minimum. For uncoded schemes, the EVT approach as compared to the threshold-based bound offers a closed-form and more accurate approximation of the average PER in block-fading channels, while latter approach is still applicable for coded schemes.

References

[1] S. Liu, X. Wu, Y. Xi, and J. Wei, “On the throughput and optimal packet length of an uncoded ARQ system over slow Rayleigh fading channels,” *IEEE Commun. Lett.*, vol. 16, no. 8, pp. 1173–1175, 2012.
[2] Y. Xi, A. Burr, J. Wei, and D. Grace, “A general upper bound to evaluate packet error rate over quasi-static fading channels,” *IEEE Trans. Wireless Commun.*, vol. 10, no. 5, pp. 1373–1377, 2011.
[3] J. Wu, G. Wang, and Y. R. Zheng, “Energy efficiency and spectral efficiency tradeoff in type-I ARQ systems,” *IEEE J. Selected Areas in Commun.*, vol. 32, no. 2, pp. 356–366, 2014.
[4] G. Song and Y. Li, “Asymptotic throughput analysis for channel-aware scheduling,” *IEEE Trans. Commun.*, vol. 54, no. 10, pp. 1827–1834, 2006.
[5] J. Galambos, *The asymptotic theory of extreme order statistics*, 1st ed. Wiley, 1978.
[6] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 7th ed. Elsevier, 2007.
[7] M. K. Simon and M.-S. Alouini, *Digital communication over fading channels*. John Wiley & Sons, 2005.

[8] M. Wu, X. Lin, and P.-Y. Kam, "New exponential lower bounds on the Gaussian Q-function via Jensen's inequality," in *IEEE 73rd Veh. Tech. Conf. (VTC Spring)*, 2011.

[9] P. O. Börjesson and C.-E. W. Sundberg, "Simple approximations of the error function Q (x) for cummunications applications," *IEEE Trans. Commun.*, vol. 27, no. 3, pp. 639–643, 1979.

[10] Y. Isukapalli and B. D. Rao, "An analytically tractable approximation for the Gaussian Q-function," *IEEE Commun. Lett.*, vol. 12, no. 9, pp. 669–671, 2008.