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Preformed Cooper pairs in flat-band semimetals

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We study conditions for the emergence of the preformed Cooper pairs in materials hosting flat bands. As a particular example, we consider a semimetal, with a pair of three-band crossing points at which a flat band intersects with a Dirac cone, and focus on the s-wave intervalley pairing channel. The nearly dispersionless nature of the flat band at strong attraction between electrons promotes local Cooper pair formation so that the system may be modeled as an array of superconducting grains. Due to dispersive bands, Andreev scattering between the grains gives rise to the global phase-coherent superconductivity at low temperatures. We develop a mean-field theory to calculate transition temperature between the preformed Cooper pair state and the phase-coherent state for different interaction strengths in the Cooper channel. The transition temperature between semimetal and preformed Cooper pair phases is proportional to the interaction constant, the dependence of the transition temperature to the phase-coherent state on the interaction constant is weaker.

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In condensed-matter systems the nearly dispersionless flat-band electronic structure may stimulate the interaction-induced instabilities. Of particular interest is the interplay between flat band and superconductivity. The reason for that is the relatively large value of the superconducting transition temperature, which can be linearly proportional to the pairing interaction strength as was proposed by Khodel’ and Shaginyan [1] and later studied for instance in Refs. [2–7].

The examples of flat-band systems include multilayer graphene with rhombohedral stacking [4], interfaces between the domains in graphene with Bernal stacking order [8], twisted bilayer graphene [9,10], and semimetals with integer pseudospin quasiparticles [11–13]. The latter is characterized by the existence of multiple-band crossing points at which the flat band intersects with the Dirac cones. For example, the low-energy electron excitations can be described by the Hamiltonian for a pseudospin-one particle, see Ref. [13].

Recently superconductivity has been observed in twisted bilayer graphene [14] and in Bernal bilayer graphene subjected to applied perpendicular electric field [15]. Signatures of superconductivity have been observed in highly oriented pyrolytic graphite [8,16]. Although, the semimetals hosting three-band touching points (and among them CoSi and RhSi) have been discovered [17–19] (see Ref. [20] for a review) and several flat-band enhanced Cooper pairing channels have been explored theoretically [5,21], superconductivity has not yet been detected. Despite intensive research, the role of flat band in the Cooper pairing is far from being understood [16].

We emphasize that the effect of the flat band on the formation of superconductivity can be twofold. On one hand, the strong enhancement of the electronic density of states leads to higher critical temperatures of Cooper pairing. On the other, its nearly dispersionless nature can be a serious impediment to pair condensation. The flat band favors the localization of quasiparticles, which suppresses the superconducting phase stiffness. It works against the long-range coherence leading rather to a situation with preformed Cooper pairing [22].

The problem of flat-band induced correlations between the Cooper pairs was analysed in Ref. [23]. The flat-band contribution to superconducting phase stiffness was shown to be finite and originate from the position-dependent matrix structure of the respective wave function. It is now believed that this contribution might eventually support the pair condensation. However, we argue that Ref. [23] overlooks superconductivity and deals with the preformed Cooper pair phase and the properties of the local pairs. In this theory, the flat-band contribution to phase stiffness results in narrow-range spatial correlations on the scale of the size of the preformed Cooper pair itself. Instead, we expect different situation, in which local Cooper pairs coexist with the Fermi liquid.

In our context, however, the Cooper pair formation and their condensation occur at different temperatures [24,25]. In contrast to the previous research [1–7], we emphasize the importance of both localized and delocalized quasiparticles on the emergence of superconductivity. In addition to the flat band, materials inevitably host dispersive bands as well, which essentially contribute to the pair condensation. Such situation exists in considered three-band semimetal. We note that our theory might be extended to explain superconductivity in bilayer graphene with twisted and Bernal stacking.

We show that with the increase of electron-electron attraction, the system reaches a state, which can be modeled by an emergent granularity. It can be described by the Cooper pairs localized inside the grains lacking the long-range coherence. The superconducting order parameter exhibits strong spatial fluctuations. The long-ranged Andreev coupling between the grains, thanks to the contribution of dispersive bands, establishes a coherent state at a lower temperature. We develop
a mean-field theory to calculate the temperatures of Cooper pairs formation and their consecutive condensation.

**Model of semimetal.** We consider a time-reversal symmetric semimetal with a pair of three-band crossing points at momenta $\pm \mathbf{k}_0$ in the first Brillouin zone as shown schematically in Fig. 1. As we ignore the single-particle intervalley scattering processes, the model Hamiltonian can be represented via a sum of two independent contributions from two valleys [13]: $\mathcal{H} = \int_\mathbf{k} \sum_{\nu, \pm} \Psi_{\mathbf{k}, \nu}^\dagger \mathbf{v}_F \cdot \mathbf{k} \Psi_{\mathbf{k}, \nu}$, where $\mathbf{v}_F$ is the Fermi velocity, $\mathbf{k}$ is the momentum measured relatively to the $\pm \mathbf{k}_0$ with $k \ll K_0$, $\int_\mathbf{k}(...) \equiv \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} (\ldots)$, and $\mathbf{S} = (S_x, S_y, S_z)$ are the Gell-Mann matrices acting on “which band” pseudospin degree of freedom; see the Supplemental Material [26]. The electron operators are defined by $\Psi_{\mathbf{k}, \nu} = [\Psi_{\mathbf{k}, +, \nu}, \Psi_{\mathbf{k}, 0, \nu}, \Psi_{\mathbf{k}, -, \nu}]^T$, where indices $\pm 1, 0$ correspond to three different bands, two of which are dispersive, $E_{\pm} = \pm \mathbf{v}_F k$, and another is flat, $E_0 = 0$. The latter is considered in the infinite mass limit approximation, so that higher order momentum corrections are neglected. We will be using $\hbar = k_0$ in 1 units throughout the paper.

To analyze superconducting instability in the system, we introduce electron Green’s function in Matsubara representation $G(\mathbf{r}, i\omega) = \int_\mathbf{k} G(\mathbf{k}, i\omega) e^{i\mathbf{kr}}$, where $\omega = (2n + 1) \pi T$ is the Matsubara frequency at temperature $T$. The Green’s function $G(\mathbf{k}, i\omega) = (i\omega - \mathbf{v}_F \mathbf{k} \cdot \mathbf{k} + \mu)^{-1}$ can be expressed as [26]

$$G(\mathbf{k}, i\omega) = \frac{1}{i\omega + \mu} \left( \frac{(\mathbf{S}_n)^2}{\omega} + \frac{1}{2} \sum_{s = \pm 1} (\mathbf{S}_m)^2 + s (\mathbf{S}_n) \right),$$  

where now $\mathbf{n}_r = \mathbf{r}/r$ is the unit vector in coordinate space and $\delta(\mathbf{r})$ is the Dirac delta function in three dimension. The second dipole-like term decays as a cube of distance smearing the delta function. In the limit of $r \to 0$ the Green’s function is cut by the interatomic distance. We also note that the spatial and frequency dependent parts are separated in the flat-band model in the infinite mass approximation.

It suffices to consider the nonlocal term in the limiting case, where $\mu \gg |\omega|$ and $\mu r/\nu_F \gg 1$,

$$G_{nl}(\mathbf{r}, i\omega) = -\frac{\mu (\mathbf{S}_n)}{4\pi \nu_F r} \left[ \operatorname{sgn} \omega + (\mathbf{S}_r) \right] e^{-\frac{\mu}{\nu_F} (\omega - i\mu) \operatorname{sgn} \omega}. \tag{3}$$

The expected three-dimensional spatial coordinate dependence is supplemented by the unusual matrix structure. Let us now discuss superconductivity in flat-band semimetal.

**Model of superconductivity.** We consider s-wave superconducting instability in the flat-band semimetal taking three-band semimetal as a particular example [5]. However, we note that our results are generally valid for systems with coexisting dispersive and nearly flat bands. The symmetry analysis of the superconducting channels in three-band semimetal was performed in Refs. [5,21]. Specifically, for clean systems possessing time-reversal symmetry, it was found that the flat band enhances intervalley Cooper pairing with total pseudospin $S = 0$. The intervalley contribution to the interaction between electrons is given by [26]

$$U = -\lambda \sum_{\alpha, \beta} \int_\mathbf{k, k'} (\Psi_{1, \alpha, \mathbf{k}}^\dagger \Psi_{1, \beta, \mathbf{k}}) (\Psi_{-1, \beta, -\mathbf{k}}^\dagger \Psi_{-1, \alpha, -\mathbf{k}}), \tag{4}$$

where $\lambda > 0$ is the interaction constant. We seek for the case in which flat-band significantly contributes to superconductivity. Among many possible superconducting states we focus on the s-wave intervalley odd pairing [5,21]. The pairing channels can be distinguished by the total pseudospin $S$ of Cooper pairs. In our case, one can only have even $S = 0$ and $S = 2$ due to the Pauli principle. We focus on the $S = 0$ channel, which has the highest superconducting transition temperature. The extended analysis of superconducting states for $S = 2$ in the model, which takes into account quadratic momentum corrections to the single-particle Hamiltonian, can be found in [27].

Let us now qualitatively estimate the superconducting vertex part describing Cooper instability,

$$\operatorname{det}[1 - \lambda \Pi(\mathbf{q})] = 0, \tag{5}$$

where $\Pi(\mathbf{q}) = T \sum_{\omega} \int_\mathbf{k} G(\mathbf{k}, i\omega) [G(\mathbf{q - k, -i\omega})]_{S-S}$ and summation is performed over the Matsubara frequencies. Due to the local term in Green’s function (2), the integrand in $\Pi(\mathbf{q})$ diverges at large wave vectors. And it is convenient to single out nonlocal contributions (3), which contain the usual logarithmic ultraviolet cutoff. All in all, we separate local and nonlocal contributions $\Pi(\mathbf{q}) = \Pi_{loc}(\mathbf{q}) + \Pi_{nl}(\mathbf{q})$ and neglect crossed terms between them (as we are interested in the two limiting cases only).

Consider momentum expansion of the vertex part $\Pi(\mathbf{q}) \approx \Pi + q^2 \delta \Pi$, where the second term describes superconducting stiffness. The contribution of the local term in Green’s function (2) to the superconducting vertex part $\Pi_{loc} \sim (K^2/\mu^4)(\mu/2T)$ is proportional to the volume of the flat band.
in momentum space $K^3 \equiv \int d^3k$. Using (3) and assuming $\mu \gg |\omega|$, a straightforward calculation results in the nonlocal term $\Pi_{\text{nl}} \sim \mu^2 \ln(\mu/T)/v_F^3$. Let us compare two terms at $\mu > T$,

$$\Pi_{\text{loc}} \sim (K/\mu)^3 \sim \frac{1}{V_G p_F^3},$$

(6)

where we introduced an effective volume $V_G \sim K^{-3}$. Let us compare the momentum dependent corrections to the vertex parts as well. Taking $\delta \Pi_{\text{loc}} \sim (K/\mu)\ln(\mu/T)$ and $\delta \Pi_{\text{nl}} \sim \mu^2/(T^2 v_F)$, we estimate

$$\delta \Pi_{\text{nl}} \sim \left( \frac{T}{\mu} \right)^2 (V_G p_F^3)^{-1/3}. \quad (7)$$

At $K \gg p_F$, we may adopt a model of a granular system, in which each grain hosts a Cooper pair. The typical volume of the grain is of the order of $V_G$.

In this limit, at large interaction constant the local contribution $\Pi_{\text{loc}}$ determines the instability towards the Cooper pair formation. The dispersionless nature of the flat band prevents establishing global coherence in the system. It rather leads to phase fluctuations of the order parameter on the scale of the size of the grain. Although, by lowering the temperature, see Fig. 2, one may reach a situation, in which the global coherence is fulfilled by long-range coupling between the grains.

**Ginzburg-Landau functional.** To proceed, we will analyze the superconducting instability within the Ginzburg-Landau (GL) functional framework in the static approximation [25]. We assume that the semimetal can be fragmented into a matrix of grains with equal volumes $V_G$ and consider the situation in which the phase of the order parameter $\Delta_i$ (where index $i$ labels the grain) varies from grain to grain, while its amplitude is grain independent. In this model, the system is described by the Bogoliubov-deGennes (BdG) Hamiltonian $\mathcal{H} = \sum_i \int d^3r d\Phi_i^*(r) H_i(r) \Phi_i(r)$, where integration is performed over the volume of the grain $V_G$ [26],

$$H_i(r) = \left[ -i v_F \mathbf{S} \cdot \nabla - \mu \frac{\Delta_i(r)}{\Delta_i^*(r)} \right] + \frac{i v_F}{2} \mathbf{S} \cdot \nabla + \mu,$$

(8)

and the Gorkov-Nambu operator on grain $i$ is given by $\Phi_i(r) = [\Psi^T_i(r), \Psi^{*-1}_i(r)]^T$. Here a unitary operator $\gamma = e^{i \mathbf{\sigma} \cdot \mathbf{S}}$ transforms the spin-1 operators as $\gamma \Psi^T \gamma^† = -\Psi$. It resembles the antisymmetric property of the spin-matrix structure of the gap function in usual superconductors. Note that we neglect single-particle intervaley scattering processes, which results in the $6 \times 6$ matrix structure of BdG Hamiltonian (similarly to the $2 \times 2$ matrix structure reduction of the BdG Hamiltonian in usual superconductors).

In the limit of small gap function $|\Delta_i| \ll \mu$, the GL functional can be further expanded in powers of the order parameter. In this expansion, the superconducting phase stiffness consists of contributions from both local and nonlocal terms in the Green’s function (1). Although noting (7), the former is smaller compared to the nonlocal contribution, which allows us to neglect variation of the order parameter inside the grain and focus on the intergrain coupling only.

Taking into account both local (2) and nonlocal (3) contributions, the GL functional yields [26]

$$F = \sum_i F_i - \sum_{i \neq j} F_{i,j} = V_G \sum_i \left\{ a |\Delta_i|^2 + \frac{b}{2} |\Delta_i|^4 \right\} - \frac{v_{\text{il}}}{2 v_F} V_G^2 \sum_i \sum_{i \neq j} e^{-\frac{1}{\Delta_{\text{il}}^2} |r_i - r_j|^2} |\Delta_i \Delta_j| \cos(\phi_{ij}), \quad (9)$$

where $a = 3(\lambda^{-1} - \lambda^{-1}_c) \ln \frac{\mu}{\frac{T}{2}}$ and $b = 3\lambda^{-1}_c (\sin \frac{T}{4\mu} - \frac{T}{4\mu})/(4\mu^2 \chi_{\text{eff}}^2)$ are the model dependent coefficients. We consider the case when the chemical potential $\mu$ is smaller than the Debye frequency. It also suffices to introduce a critical value of interaction constant $\lambda_c = 6\mu/K^3$. The last term in (9) describes long-range Andreev coupling between the grains, which is weighted by the density of states per valley at the Fermi energy $v_{\text{il}} = \mu^2/(2\pi^2 v_F^3)$. To obtain this term one follows familiar microscopic derivation within the GL formalism [28]. Note that the Andreev term is smaller than the second term in the coefficient $a$. The latter is defined by the flat-band contribution. We neglect weak corrections from delocalized states to the coefficient $a$ within the granular model. Andreev coupling contributes to quartic terms in general form $\propto \Delta_i \Delta_j \Delta^*_i \Delta^*_j$, although these terms are small compared to $b |\Delta|^4$ in (9).

Consider a situation in which weak Andreev coupling between the grains can be neglected. At $a < 0$, from the extremum of (9), we obtain nonzero local $|\Delta_i|$ with random phase. We identify this case as preformed Cooper pair phase. In this case, equation $a = 0$ determines the temperature of preformed Cooper pair formation on the grain.

Provided $\lambda \gg \lambda_c$ one obtains $T_0 = \mu/2 \arctan(\lambda_c/\lambda)$ [5]. This is the temperature of the phase transition between a doped semimetal and preformed Cooper pair state. The
low-doping case $\mu \ll T$ requires large interaction constant $\lambda/\lambda_c \gg 1$ for the transition. Here the critical temperature is proportional to the interaction constant and inversely proportional to the volume of preformed Cooper pair $T_p = \lambda K^3/12$ [3–6].

At high doping $\mu \gg T$, the transition takes place when the interaction constant is larger than the critical value $\lambda \simeq \lambda_c$ [24]. In this limit the coefficients in (9) can be simplified as $a = 3(\lambda^{-1} - \lambda_c^{-1})$ and $b = 3/(2\mu^2\lambda_c)$. We shall focus on this case in what follows. Let us now calculate the transition temperature to the phase-coherent state, which is driven by the Andreev coupling.

Transition between preformed-pair and phase-coherent states. With the increase of interaction constant $\lambda$, the impact of dispersive bands enhances Andreev coupling between the superconducting grains. As a result, the system may reach the phase coherence. In what follows, we develop a mean-field theory to calculate the superconducting transition temperature.

Within the mean-field approach, the fluctuating values of the order parameter $\Delta$ are replaced by an average order parameter $\langle \Delta \rangle$ [25,26]. The self-consistent mean-field equation is

$$
\langle \Delta \rangle = \langle \Delta \rangle \frac{\int \Delta \Delta^* - \mu e^{-F/T}}{\int \Delta \Delta^* e^{-F/T}} \approx \langle \Delta \rangle \frac{\int d\Delta d\Delta^* e^{-F_M / T}}{\int d\Delta d\Delta^* e^{-F_M / T}},
$$

in which $\Delta \Delta \equiv \Pi d\Delta$. The mean-field functional reads

$$
F_{MF} = F_0 - \sum_{i \neq 0} F_{i0} = V_G \left[ a|\Delta|^2 + \frac{b}{2}|\Delta|^4 - c(\langle \Delta \rangle \Delta^* + \langle \Delta^* \Delta \rangle) \right].
$$

In the continuum limit, we substitute $\sum_{i \neq 0} = V_G^{-1} \int d\mathbf{r}$ and obtain coefficient $c = v_{nl} \ln |\mu / T|$ within the logarithmic accuracy.

Without loosing the generality, the averaged order parameter $\langle \Delta \rangle$ can be restricted to real value. At $c(\Delta) \ll \sqrt{|a|T}/V_G \max(1,bT/aV_G)$, expanding integrands in Eq. (10) in powers of $\langle \Delta \rangle$, we obtain

$$
1 = v_{nl} \frac{V_G \langle |\Delta|^2 \rangle}{T} \ln |\mu / T|, \quad \langle |\Delta|^2 \rangle = \int_0^\infty dx x e^{-\frac{\mu}{b}(\frac{1}{2}x^2 + \lambda^{-1} x^2)}/\int_0^\infty dx x e^{-\frac{\mu}{b}(\frac{1}{2}x^2 + \lambda^{-1} x^2)}.
$$

The solution to Eq. (12) is shown in Fig. 2. Analytical expressions can be analysed in several limiting cases.

First, consider a situation in which the interaction constant $\lambda$ is much smaller the critical value, $\lambda \ll \lambda_c$, so that $\Delta_i = 0$ ($a > 0$). In this weak-coupling regime, the $b\Delta_i$ term in formula (12) can be neglected provided $a^2 V_G / bT \gg 1$. Performing integration in (12) one obtains expression for the square of quasiparticle energy gap $\langle |\Delta|^2 \rangle \approx T/aV_G$. As a result, the transition temperature to the coherent state is given by

$$
T_c = \mu \exp \left\{ -\frac{3}{\lambda_v} \frac{v_{nl}}{2} (1 - \lambda / \lambda_c) \right\}.
$$

This mean-field solution coincides with the exact BCS expression. The flat band gives $1 - \lambda / \lambda_c$ enhancement correction in the exponent.

Second, consider a semimetal at the vicinity of the transition to preformed Cooper pair phase, $\lambda \approx \lambda_c$. At $a^2 V_G / bT \ll 1$, we can neglect $a$ term compared with the nonlinear $b$ term in (12) and obtain $\langle |\Delta|^2 \rangle \approx \sqrt{2T / \pi bV_G}$. Taking into account $V_G K^3 \approx 1$ and using expressions for $\lambda_c$ and $b$, we find

$$
T_c = \frac{2\mu}{9\pi} \left( \lambda_c v_{nl} \ln |\mu / T_c| \right)^2.
$$

This result is valid for both signs of the coefficient $a$. Due to $\lambda v_{nl} < 1$, the $T_c$ is proportional to the critical value of the interaction constant squared.

Third, consider the preformed Cooper pair phase, $\lambda \gtrsim \lambda_c$. For weak fluctuations $a^2 V_G / bT \gg 1$, using $\langle |\Delta|^2 \rangle \approx -a/b = 2\mu^2 (1 - \lambda - \lambda_c / \lambda_c)$, we obtain

$$
T_c = \frac{\lambda_c v_{nl}}{3} \mu \left( 1 - \frac{\lambda_c}{\lambda} \right) \ln |\mu / T_c|.
$$

The transition temperature increases with the increase of $\lambda$. However, in the limit of $\lambda \gg \lambda_c$, the GL expansion is no longer valid. The investigation of this case deserves a separate study.

Conclusions. Let us now briefly comment on the effect of finite $\propto k^2$ corrections to the Hamiltonian of semimetal. In this case, the flat band acquires a finite curvature. As noted in Ref. [5] accounting for such term results in vanishing of the threshold value $\lambda_c$, which is required for preformed Cooper pairing, provided the chemical potential crosses the band. Hence, enhancement of the transition temperature $T_c$ (13) at smaller values of the interaction constant $\lambda \to 0$ is expected for particular doping, which depends on the sign of $\propto k^2$ correction term. We also note that materials may contain other dispersive bands, which can coexist with the Dirac cones at the chemical potential, and contribute to the long-range coupling as well.

It would be interesting to extend the above-presented research to explain superconductivity in twisted bilayer graphene [14] and in graphite with Bernal stacking order [8]. The moiré pattern can be modeled as a system of coupled grains [29]. We argue that in this situation the intergrain coupling leads to the phase-coherent state at temperatures lower than the temperature of the on-grain Cooper pair formation. We will consider superconductivity in twisted bilayer graphene in future work.

To conclude, in this paper we have demonstrated that a nearly dispersionless flat band at strong attraction between electrons manifests itself in the emergent granularity and the Cooper pair preformation. The dispersive bands, which coexist with the flat bands, promote the global phase-coherent superconducting state at low temperatures. We have calculated the temperature of the phase transition between the preformed pairs and phase-coherent states in a semimetal hosting a pair of three-band crossing points. Experimentally, the preformed Cooper pairs may be probed locally via low-temperature spectroscopy [22].
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