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Anomalous chiral transport with vorticity and torsion: Cancellation of two mixed gravitational anomaly currents in rotating chiral $p + ip$ Weyl condensates

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Relativistic gravitational anomalies lead to anomalous transport coefficients that can be activated at finite temperature in hydrodynamic and condensed matter systems with gapless, linearly dispersing fermions. One is the chiral vortical effect (CVE), an anomalous chiral current along the system's rotation axis, expressed in terms of a gravimagnetic metric field in a rotating frame with mixed gravitational anomaly. Another one arises in the presence of hydrodynamically independent frame fields (and spin connection) and leads to the thermal chiral torsional effect (CTE). We discuss the relation of CVE, CTE, and gravitational anomalies for relativistic fermions from the perspective of nonzero torsion and the Nieh-Yan anomaly when the currents depend on the frame fields and connection instead of the metric. The transport coefficients induced by the two gravitational anomalies at zero frequency and momentum are found to be closely related and equal. At level of linear response, their difference is demarcated whether or not torsion is nonzero and the existence of nonmetric degrees of freedom in the hydrodynamic constitutive relations with sources. In particular, the relativistic anomaly from torsion is well defined, since instead of an UV divergent term the chemical potential or temperature scales enter. This is closely related to the derivation of CVE from the fourth order in gradients gravitational anomaly and its appearance already in the linear response. However, the torsional anomaly is second order in gradients and directly contributes in linear response for CTE, implying also the same for CVE. For an example where the two anomalies are sourced independently, we consider chiral $p + ip$ Weyl superfluids and superconductors rotating at finite temperature. At low energies in the linear approximation, the system is effectively relativistic along a special anisotropy axis. The hydrodynamics is governed by two velocities, the normal velocity v_n and superfluid velocity v_s . The existence of the two thermal anomalies in the condensate follows from the normal component rotation and the dependence of the momentum density on the superfluid velocity (order parameter). In the CVE, the chiral current is produced by the solid body rotation of the normal component with (angular) velocity $v_n = \Omega \times r$. In the CTE, a chiral current is produced by the vorticity of the superfluid velocity $\nabla \times v_s$, which in the low-energy quasirelativistic effective theory plays the role of gravitational torsion. In thermal equilibrium, $\langle \nabla \times v_s \rangle = 2\Omega$ spatially averaged and the two gravitational anomaly currents cancel each other. This is a version of the Bloch theorem for axial currents, prohibiting finite current in equilibrium, realized as the cancellation of two gravitational anomalies with independent sources: gravimagnetic rotation field Ω and torsion from v_s . Although the latter physically represents the superfluid vorticity similar to the CVE, in the low-energy quasirelativistic theory, it arises from torsion coupling to the normal component chiral fermions.

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I. INTRODUCTION

Consequences and implications of field theoretic quantum anomalies [1–6] in the low-energy hydrodynamic transport in chiral media have recently attracted considerable interest with applications spanning ultrarelativistic quark-gluon plasmas in heavy-ion collisions [7–11], holographic phenomenology [12–15], and topological materials with protected quasirelativistic chiral fermions including semimetals, superfluids, and superconductors [16–40].

The quantum field theoretic anomalies are in four dimensions defined by triangle diagrams with gauge and gravitational vertices, which in the low-energy hydrodynamic regime lead to the chiral magnetic (CME) and chiral vortical effects (CVE) at nonzero chemical potential and temperature [7,9,10,41,42]: The CME is a chiral current along magnetic field at finite chemical potential(s), while the CVE involves a chiral current induced by hydrodynamic fluid flow with nonzero vorticity, e.g., the angular

momentum of a rotating chiral system. Intuitively speaking, the CME is based on the analogy of (chiral) chemical potential with (chiral) temporal gauge fields, whereas the CVE rests in addition on the analogy of rotating frames to (tidal) gravitational forces.

A physically transparent way to find such transport coefficients is the requirement of positive entropy current in the presence of anomalous sources, in accordance with the second law of thermodynamics [43]. On the other hand, resting on the analogy with gravity, the appearance of the static, equilibrium anomaly transport coefficients can be organized as an order-by-order derivative expansion of the hydrodynamic conservation laws on nontrivial geometric backgrounds with sources [44–47]. The gravitational (i.e., thermal [38,48]) CVE anomaly current at linear order in response has been related to the gravitational anomalies [15,47,49] that, at least naively, appear only at higher orders in the hydrodynamic gradient expansion. The precise way in which higher-order anomalies can contribute to the transport has led to many useful insights about relativistic and nonrelativistic hydrodynamics, e.g., Refs. [50–61].

In a more recent development, there has been renewed interest in the field theoretic and hydrodynamic anomalies with nonzero torsion (and relatedly, so-called pseudo-gauge-fields) in systems with independent frame fields and connection, in contrast to just the metric. Several authors have considered nonzero torsional transport coefficients, given some assumptions beyond standard relativistic field theory and/or finite chemical potential and temperature [62–86]. The independent gravitational Nieh-Yan (NY) anomaly term from torsion [87–95] is second order in the gradients, in contrast to the fourth-order mixed R^2 gravitational anomaly from topological anomaly polynomials, and appears with a dimensionful scale parameter that is, apparently, nonuniversal and unquantized. With considerable past and more recent history, many authors have argued for and against this term with varying and contrasting results, especially whether a dimensionful UV scale can appear in the anomaly coefficient. Here, we are content to remark that without extra UV completion this term can be removed in relativistic Weyl/Dirac systems, since there is a local counterterm available that breaks no additional symmetries, while *a priori* such a term is possible in nonrelativistic systems with a cutoff scale to low-energy chiral transport and fermions. The experimentally verified chiral anomaly in $^3\text{He-A}$ can be matched consistently with the torsional anomaly both at zero and finite temperature [69,74]. See Appendix B for more discussion.

Notwithstanding, both the relativistic and nonrelativistic torsional anomalies can appear with the (usually) IR temperature or chemical potential scales, similarly to the gravitational contribution to the CVE, and we focus here on these terms. Moreover, such a finite temperature (or chemical potential) term can be “universal” as it rests on the anomaly related quantum statistical properties, with the same caveats

as the hydrodynamic CVE (or CME). In this paper, we discuss the chiral torsional effect (CTE) from the perspective of the analogy and similarity with CVE in relativistic systems with chiral fermions. We find that the coefficient is nonzero and directly related to that of the CVE (at the level of linear response) and in accordance with the previous results found with, e.g., the simple Landau level approach to chiral torsional anomalies [62,71]. In particular, recent papers [38,80] finding no torsional contribution can be easily incorporated with our framework by noting that when computing the linear response around flat space, torsion has been actually set to zero with only purely metric variations. Nevertheless, we find that nonzero transport coefficients in the case torsionful sources do appear. This discrepancy is equivalent to whether or not independent frame fields (i.e., torsion) appear in the hydrodynamics and need to be separately addressed at the level of the constitutive relations in the presence of sources.

Although a relativistic system (or gravity [96]) with torsion remains to be identified in the real world, we note that geometric torsion has been found to be inherent in the hydrodynamics of many condensed matter systems, including elasticity (see, e.g., Refs. [97–102]), topological quantum Hall and paired systems [69,103,104], semimetals [62–64,67,71], crystalline insulators [105,106], and in general nonrelativistic Newton-Cartan spacetimes [107–109]. For an example of the difference and similarities of the two independent anomaly sources, we discuss the two anomalies in a rotating chiral (nonrelativistic) $p + ip$ Weyl superfluid or superconductor with a normal component and vortex lattice with vorticity and low-energy effective torsion, respectively. This gapless system is not strictly relativistic [110] but instead, at low energies, described by spatially anisotropic Newton-Cartan geometry that identifies the independent hydrodynamic variables and symmetries directly. However, in the linear approximation, the low-energy theory satisfies effective (local) Lorentz invariance along the special anisotropy direction; therefore, the ingredients for the relativistic CVE and CTE can be applied if the rotation, vorticity, and torsion are along this special axis.

The rest of this paper is organized as follows. In Sec. II, we review the CVE and chiral anomaly. In Sec. III, we discuss the gravitational anomaly perspective of CVE and CTE. Section IV presents the Kubo formula argument for CTE and its relation to CVE. Section V discusses the cancellation of CVE and CTE in rotating chiral $p + ip$ Weyl condensates. Conclusions and the outlook end the paper, with an Appendix containing geometric conventions and formulas.

II. CHIRAL ANOMALY, CHIRAL MAGNETIC, AND VORTICAL EFFECTS

To set the notation and introduce the anomalous currents, we consider the low-energy, effective theory of relativistic Dirac fermions in a chemical potential of a low-energy hydrodynamic fluid, $\gamma^0\mu \rightarrow \gamma^\mu\mu u_\mu$ [51],

$$S_{\text{hydro}}[\psi, \psi^\dagger] = \int d^4x \bar{\psi} \gamma^\nu [i(\partial_\nu - iqA_\nu) + \mu u_\nu] \psi. \quad (1)$$

Here, u_μ is a local velocity field for a fluid element in its rest frame, $u^\mu u_\mu = 1$, and is a generalized chemical potential for the fermions. As is obvious, u_μ enters similarly as a U(1) gauge field,

$$\frac{\delta S_{\text{hydro}}}{\delta u_\mu} = \mu \bar{\psi} \gamma^\mu \psi = \mu J^\mu. \quad (2)$$

Relativistically, the u_μ couples to μJ^μ with units of (energy) momentum. The fermions are massless and cannot equilibrate with the low-energy fluid velocity u_μ ; the presence of the low-energy background fluid u_μ singles out a preferred frame. Similarly, one can introduce an axial chemical potential velocity field $\mu_5 u_{5\mu} \gamma^5$.

From now on, we focus on single right-handed fermion $\chi = +1$ with a global (and local) U(1) and set $\mu = \mu_5 = \mu_\chi/2$. Calculating the anomalous current of (1), loosely via the implied substitution $qA_\mu \rightarrow qA_\mu + \mu u_\mu$, we arrive at

$$J_\chi^\mu = \frac{\epsilon^{\mu\nu\lambda\rho}}{8\pi^2} (eA_\nu + \mu u_\nu) (e\partial_\lambda A_\rho + \mu\partial_\lambda u_\rho). \quad (3)$$

The cross-terms give the CME, e.g., in the simplest case when $u_\mu = (1, 0, 0, 0)$. Setting $A_\mu = 0$, we obtain the chiral vortical effect (at $T = 0$ and to lowest order in μ)

$$J_\chi^\mu = c_\chi \frac{\mu_\chi^2}{8\pi^2} \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho, \quad (4)$$

where $c_\chi = \chi = +1$ is the normalized coefficient of the chiral U(1) anomaly for right-handed fermions. Hence, heuristically, the coefficients of the chemical potential term of the CVE and CME are implied by the chiral Adler-Bell-Jackiw anomaly.

III. MIXED GRAVITATIONAL AND TORSIONAL ANOMALIES

On the other hand, we can consider gravitational action of Dirac fermions, understood to be related to an effective action in the hydrodynamic regime around a near-equilibrium background with sources [44–47],

$$S_{\text{grav}}[\psi, \bar{\psi}] = \int d^4x e \bar{\psi} e_a^\mu \gamma^a i \left(\partial_\mu - \frac{i}{2} \omega_{\mu ab} \gamma^{ab} \right) \psi, \quad (5)$$

where $g_{\mu\nu} = e_\mu^a e_\nu^b \eta^{ab}$ in terms of the tetrad e_μ^a , with inverse e_a^μ , and $\omega_{\mu ab} \gamma^{ab}$, $\gamma^{ab} = \frac{i}{4} [\gamma^a, \gamma^b]$, is the spin connection in the spin $1/2 \oplus 1/2$ Weyl representation. We consider two cases here: when $S_{\text{eff}} = S_{\text{grav}}[g_{\mu\nu}]$ depends only on the metric $g_{\mu\nu}$ and when $S_{\text{eff}} = S_{\text{grav}}[e^a, \hat{\omega}_\mu]$ depends on e_μ^a and $\hat{\omega}_\mu$ independently. To compare with S_{grav} with S_{hydro} , we set $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is small perturbation,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - 2\mathbf{u} \cdot d\mathbf{x} dt - d\mathbf{x}^2, \quad (6)$$

and $-\mathbf{u}_i = h_{ti}$ is a small velocity. In terms of tetrads/vierbein $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$,

$$\begin{aligned} e_\mu^0 &= (1, -\mathbf{u}), & e_\mu^m &= \delta_\mu^m \\ e_0^\mu &= (1, 0), & e_m^\mu &= (u_m, \delta_m^\mu). \end{aligned} \quad (7)$$

The torsion-free spin connection corresponding to this variation is collected in the Appendix C. To conclude, from this hydrodynamic variation, we compute correlation functions of the energy-momentum tensor,

$$\delta S = \int d^4x e \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu}, \quad (8)$$

where $T^{\mu\nu} = \frac{1}{2} (e_a^\mu T^{a\nu} + e_a^\nu T^{a\mu})$ is the symmetric energy-momentum tensor and $\delta g_{\mu\nu} = e^{a\mu} \delta e_a^\nu + e^{a\nu} \delta e_a^\mu$ is the symmetric variation. We have assumed that the Lorentz anomaly vanishes and T_μ^a can be made symmetric; see Sec. VI for more discussions.

With nonzero torsion, the tetrad, connection, and variations are different. See Appendix A for our conventions regarding torsion. In terms of variations [108],

$$\begin{aligned} \delta S &= \int d^4x e [\tilde{T}_a^\mu \delta e_\mu^a + s^\mu_{ab} \delta \omega_\mu^{ab}] \\ &= \int d^4x e [T_a^\mu \delta e_\mu^a + s^\mu_{ab} \delta K_\mu^{ab}] \\ &= \int d^4x e [T_a^\mu \delta e_\mu^a + S_a^{\mu\nu} \delta T_{\mu\nu}^a], \end{aligned} \quad (9)$$

where

$$\tilde{T}_a^\mu = \frac{1}{e} \frac{\delta S}{\delta e_\nu^a}, \quad s^\mu_{ab} = \frac{1}{e} \frac{\delta S}{\delta \omega_\mu^{ab}} \quad (10)$$

are the tetrad energy-momentum and (intrinsic) spin currents. The $K_{\mu b}^a = (\omega - \hat{\omega})_{\mu b}^a$ is the contorsion tensor; $\hat{\omega}_\mu$ is the Christoffel connection fully determined by e_μ^a ; and, finally, $S_a^{\mu\nu} = \frac{1}{2} \eta_{ab} e_\lambda^b (s^{\mu\nu\lambda} - s^{\nu\lambda\mu} - s^{\lambda\mu\nu})$. The \tilde{T}_μ^a and T_μ^a differ by torsion and spin-current terms [108]. The different variations arise whether e^a and ω^{ab} are treated as independent, the e^a and K^{ab} or, finally, e^a and T^a . We shall see below that around flat space the different variations of (9) are simply related.

A. Gravitational anomaly and CVE

The hydrodynamic effective action S_{grav} contains only equilibrium chemical potentials and gravitational (geometric) background fields [47]. Now, to the lowest order in the absence of torsion, the mixed chiral-gravitational anomaly is given as

$$\nabla_\mu J_\chi^\mu = \frac{d_\chi}{768\pi^2} e^{\mu\nu\lambda\rho} R_{\beta\mu}^\alpha R_{\alpha\lambda\rho}^\beta. \quad (11)$$

For single right-handed chiral fermion with global $U(1)$, $d_\chi = 1$. Rather unexpectedly, the chiral vortical effect and mixed gravitational anomaly (11) share the same coefficient which is not renormalized by fermionic loops in free and many interacting or holographic theories; see, e.g., Refs. [9,15,41,47,50,55,111] and references therein. Namely, utilizing the metric variations (6), (8), one can show that the chiral vortical effect [41] contains the term

$$J_\chi^\mu = d_\chi \frac{T^2}{24} \omega^\mu, \quad \omega^\mu = \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho, \quad (12)$$

where ω^μ is the local fluid vorticity and d_χ is the mixed anomaly coefficient. We stress that the general form of the CVE can be derived without recourse to the gravitational anomaly Eq. (11), see e.g., Refs. [9,41,43,44,50,58,112]. Nevertheless, the precise value of T^2 the coefficient seems to be related to the gravitational mixed and global anomalies [15,47,50,59,60,111]. Kinematic third-order terms directly from (11) were identified recently [113].

Here, we will see that the Nieh-Yan gravitational anomaly directly provides a T^2 term with the same coefficient (up to higher-order terms) from the CTE, provided we allow for background torsion. Above, we already discussed the anomalous current $\propto c_\chi \mu^2$ induced by a chemical potential from the chiral anomaly with coefficient c_χ . Below in Sec. IV, the emergence of the term $\propto d_\chi T^2$ with coefficient d_χ is discussed at the level of the linear response Kubo formula for free fermions around flat space with and without torsion.

B. Nieh-Yan gravitational anomaly and CTE

On spacetimes with torsion, there is also the CTE [72,73] from the mixed chiral-torsional anomaly (thermal Nieh-Yan anomaly [74–77]; see Appendix B):

$$J_\chi^\mu = -t_\chi \frac{T^2}{48} e^{\mu\nu\lambda\rho} e_\nu^0 T_{\lambda\rho}^0. \quad (13)$$

Here, $t_\chi = d_\chi = 1$ for a right-handed fermion. In linear response, $J_\chi^k = t_\chi \frac{T^2}{48} \epsilon^{kij} T_{ij}^0$ in analogy with the CVE as we discuss below at the level the Kubo formula with torsion.

Moreover, the connection to CVE follows in the simplest terms via the identification $u_\mu = u_a e_\mu^a = e_\mu^0$, where $u_a = (1, 0, 0, 0)$, whence the CVE becomes

$$\epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho = \epsilon^{\mu\nu\lambda\rho} u_a u_b e_\nu^a \partial_\lambda e_\rho^b = \epsilon^{\mu\nu\lambda\rho} e_\nu^0 \partial_\lambda e_\rho^0 \quad (14)$$

by assuming nonzero timelike torsion. Relativistically, $u_\mu u^\mu = u^a u_a = 1$, and in the fluid rest frame, $u_a = (1, 0, 0, 0)$ at equilibrium. We note that nonrelativistically one usually fixes the timelike covector $u_a = (1, 0, 0, 0)$

with the Newtonian clock form, whereas the velocity $u^a = (1, \mathbf{u})$ is independent with spatial vorticity $\nabla \times \mathbf{u}$.

IV. TORSIONAL KUBO FORMULA AND CURRENTS

Building on the earlier and recent work, let us present the Kubo formula for the CVE and derive from that its version with nonzero torsion.

A. Kubo formula with torsion

The CVE current is

$$J_\chi^\mu = \sigma_\chi^V e^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho \quad (15)$$

where σ_χ^V is the chiral vortical conductivity. For applications, we are interested in the axial current $J_5 = \sum_\chi \chi J_\chi = J_+ - J_-$. Since $u_\lambda = h_{t\lambda} = (1, u_i)$, its Kubo formula reads (no sum over m) [15]

$$\sigma_\chi^V = \lim_{k \rightarrow 0} \frac{i\epsilon_{ijm}}{2k^m} \langle J_\chi^i T^{tj} \rangle(k)_{\omega=0} \quad (16)$$

where $\delta S = \int d^4x \sqrt{-g} T^{t\lambda} h_{t\lambda}$. This is evaluated as

$$\begin{aligned} \sigma_\chi^V &= \lim_{k \rightarrow 0} \frac{i\epsilon_{ijm}}{2k^m} \\ &\times \int d^4x e^{ik \cdot (x-x')} \theta(t-t') \langle [J_\chi^i(x), T^{tj}(x')] \rangle_{\omega=0}. \end{aligned} \quad (17)$$

This correlator was computed in Ref. [15] for free fermions. It is composed of two terms, $\sigma_\chi^V = \sigma_{(0j)}^{\tilde{T}} + \sigma_{j0}^{\tilde{T}}$, where we defined the torsional conductivity $\sigma_{(a\mu)}^{\tilde{T}} = \lim_{k \rightarrow 0} i\epsilon_{ijm} (k^m)^{-1} \langle [J_\chi^i, \tilde{T}^{a\mu}] \rangle_{\omega=0}$. This follows directly from

$$\begin{aligned} T^{tj} &= \frac{i}{2} \bar{\psi} (\gamma^0 \partial^j + \gamma^j \partial^t) \psi \\ &= \frac{1}{2} (e_0^t \tilde{T}_i^0 g^{ij} + e_m^j \tilde{T}_t^m g^{tt}) \\ &= \frac{1}{2} (\tilde{T}^{0j} + \tilde{T}^{jt}). \end{aligned} \quad (18)$$

We make no distinction between the indices, as in linear response $e_a^\mu = \delta_a^\mu$, $g^{\mu\nu} = \eta^{\mu\nu}$, and

$$\tilde{T}^a{}_\mu := \frac{1}{e} \frac{\delta S}{\delta e_a^\mu} \Big|_{\hat{\omega}_\mu} = -e_\mu^b e_a^\nu \tilde{T}_b^\nu. \quad (19)$$

When torsion vanishes, we should use T^{0j} corresponding to the metric variation $\delta g_{\mu\nu} = h_{\mu\nu}$, while for nonzero torsion, we should use \tilde{T}_a^μ and δe_a^μ . We now focus explicitly on the latter case with torsion.

The vacuum subtracted, finite result for both static correlators involving \tilde{T}^{0j} and \tilde{T}^{ji} turns out to be identical [15,41] as $k \rightarrow 0$,

$$\begin{aligned}\sigma_{(0j)}^{\tilde{T}} &= \sigma_{(ji)}^{\tilde{T}} = \frac{1}{8\pi^2} \int_0^\infty dq q [n_F(q + \mu) + n_F(q - \mu)] \\ &= \frac{\mu_\chi^2}{8\pi^2} + \frac{T^2}{24},\end{aligned}\quad (20)$$

where $n_F(x)$ is the Fermi distribution. From the definitions of \tilde{T}_μ^a and \tilde{T}_b^ν , evaluated on the background (7) with nontrivial e_μ^0 , this translates to

$$J_\chi^\mu = \sigma_{(0i)}^{\tilde{T}} \epsilon^{\mu\nu\lambda\rho} e_\nu^0 \partial_\lambda e_\rho^0 = \frac{\sigma^{\tilde{T}}}{2} \epsilon^{\mu\nu\lambda\rho} e_\nu^0 T_{\lambda\rho}^0, \quad (21)$$

where $\sigma^{\tilde{T}} = c_\chi \frac{\mu_\chi^2}{8\pi^2} + d_\chi \frac{T^2}{24}$. We note that precisely the same distribution integral for $\sigma_{(a\mu)}^{\tilde{T}}$ as in (20) was found utilizing the Landau level approach, there just with nontrivial spatial e_μ^m at finite temperature [71,74,76].

In hindsight, the same conclusion for CTE follows almost trivially from the CVE by setting $e_i^0 = u_i$ and the assumption that torsion is nonzero (i.e., $\hat{\omega}_\mu = 0$). This argument was first utilized in Ref. [72]. The main result of Ref. [15] is that

$$\sigma_5^V(\mu = \mu_5 = 0) = \text{Tr}(T_A)_R - \text{Tr}(T_A)_L = d_A \quad (22)$$

is the gravitational anomaly coefficient for N_f fermions with T_A the generators of a global symmetry G . For a pair of $U(1)_\chi$ Weyl fermions with opposite chirality $\chi = \pm 1$, we recover the axial $\sigma_5^{\tilde{T}} = \sigma_5^V = \frac{\mu^2 + \mu_5^2}{4\pi^2} + \frac{T^2}{12}$; see below.

More recently, this result was discussed in Ref. [80], however solely in terms of the CVE in (17). Nevertheless, the nonzero quantities $\sigma_{(a\mu)}^{\tilde{T}}$ in the correlation function of the Kubo formula show explicitly that on torsional backgrounds the NY form contributes to the current with the same coefficient. In linear response around flat spacetime, the chiral vortical and torsional responses are simply related with distinction only in the chosen hydrodynamic sources (metric vs tetrad and connection). If torsion vanishes and \tilde{T}_μ^a is symmetric, the CVE formula is recovered. This is equivalent to that only $T^{\mu\nu}$ and the metric $g_{\mu\nu}$ enter the response or more generally the hydrodynamic constitutive relations with sources. In contrast, on torsional backgrounds, \tilde{T}_a^μ and e_μ^a (or related quantities) are the appropriate hydrodynamic variables and sources.

This completes our review of the Kubo formula with (and without) torsion. Next we discuss these results in comparison to the arguments of Ref. [80], which in contrast to this paper did not find any currents from torsion.

B. Chiral torsional currents

The above Kubo formula can be used to compute the response to torsion. We now clarify its relation to other, independent derivations of anomalous gravitational and gauge currents. Following Ref. [80], let us do nonzero variations of torsion in two independent ways around flat space: (i) first, varying the spin connection but holding the tetrad fixed, and, (ii) second, keeping the spin connection fixed, varying only the tetrad. These variations correspond to the first two torsionful variations in (9). Throughout, we will keep all other sources absent, so that the current is zero only if torsion does not contribute. The two variations (i) and (ii) are as follows.

- (i) In this case, $\delta\omega$ is the only nonzero variation. Since the tetrad is fixed, the spin connection is exclusively (con)torsional and contributes through an axial term from totally antisymmetric torsion; see, e.g., Ref. [93].

In more detail, the NY form is given as $\delta(e_a \wedge T^a) = e_a \wedge \delta T^a = e_a \wedge \delta\omega_b^a \wedge e^b$, with $\delta\omega$ contributing as effective axial gauge field,

$$\nabla_\mu = \partial_\mu - i\gamma^5 \frac{1}{8} \epsilon_\mu^{\nu\rho\lambda} \delta T_{\nu\lambda\rho} \equiv \partial_\mu + i\gamma^5 \delta A_{5\mu}^T, \quad (23)$$

where the lhs defines the rhs and the superscript “ T ” in δA_5^T reminds us that this axial field originates from the variation of (con)torsion, *not* a gauge field or chemical potential. In particular, for spatial antisymmetric torsion, the result is that torsion contributes to the current $\star J_5 = \sigma^T e_a \wedge T^a$ with the coefficient,

$$\begin{aligned}J_5^t &= \sigma_5^T \delta A_5^{Tt} \\ &= \frac{1}{8} \sigma_5^T \delta(\epsilon^{tijk} \delta T_{ijk}) \\ &= -\frac{1}{4} \sigma_5^T \star(e_m \wedge \delta T^m)^t,\end{aligned}\quad (24)$$

where the last line defines the one form dual of $e_a \wedge T^a$. The current is *exclusively* from the torsional NY term and vanishes only if torsion is zero. With the relations (23) and (24), the coefficient is found easily using the result

$$\frac{1}{4} \sigma_5^T = \frac{1}{4} \langle J_5^t J_5^t \rangle = \frac{1}{4} \frac{d\rho}{d\mu_5^T} = \frac{\mu^2 + \mu_5^2}{4\pi^2} + \frac{T^2}{12}. \quad (25)$$

This is equal to (20) with spacelike torsion. In contradistinction with Eqs. (24) and (25), the quantity $\frac{d\rho}{d\mu_5^T}$ was not included in coefficients $c_{T\perp}$ in Eq. (21) of Ref. [80]. In addition, the various coefficients in Eqs. (28)–(33) of Ref. [80] are different than (23) and contain zeroth-order variations from u_μ . But the current with coefficient equal

to the charge susceptibility $\frac{d\rho}{d\mu_5}$ arises solely from torsion in the absence of other sources. See also Ref. [83] and the more recent discussion in Ref. [84].

Of course, the fact that variation of $\delta\mu_5 \sim \delta A_5^t$ is related to torsion follows from (23), and this is the reason that the transport coefficient σ_5 with chiral anomaly U(1) contribution arises. But this was merely due to a convenient way to compute the response: the background variation of J_5^t for $\delta A_5^t \sim \mu_5$ with gauge fields is physically distinct from the (con)torsionful variation $\delta\omega_b^a \sim \delta T^a$. In the former, the background has no torsion, just a source of A_5 , and in the latter, nonzero torsion is the only nonvanishing source. In particular, there is no vorticity $du = 0$, since $\delta u_a = \delta e^a = 0$. From the computation of CVE, the two responses are equal at the level of the linear response, and the gravitational anomaly term $\propto T^2$ is “explained” by the contorsionful spin connection $\delta\hat{\omega}$. We remark that when both μ, μ_5 are nonzero the hydrodynamics for a consistent (gauge-invariant) theory need to be solved separately, but the problematic vector currents are distinct from J_5 above.

- (ii) Now, we discuss the second independent linear response variation with torsion. Here, we take the 1-form $e^0 = e_\mu^0 dx^\mu$. In general, e^a (or e_a) are varied with everything else *fixed*; in particular, $\hat{\omega}_\mu = 0$ and constant $u^a = (1, 0, 0, 0)$.

These conditions set $\delta T^0 = de^0 \neq 0$. Now, $u = u_a e^a = u_0 e^0$. Then, following Refs. [72,80],

$$\begin{aligned} \star J_5 &= c_v u \wedge du = c_v (u_0 e^0) \wedge d(u_0 e^0) \\ &= c_v u_0^2 e^0 \wedge de^0 = c_v e_0 \wedge de^0 = c_v e_0 \wedge \delta T^0. \end{aligned} \quad (26)$$

Again, one should assign this term exclusively to the background vierbein; it is linear in torsion, and variation δu changes as function of δe_i^0 only. This corresponds the result originally by Khaidukov and Zubkov [72]. The consistent choice is $\delta u_a = 0$ in order to compute the exclusively torsional response probing nonzero δT^0 ; in the notation of Ref. [80], the argument gives simply $c_{T\parallel} = c_v$ but not “ $c_{T\parallel} = 0$.”

The above variation (ii) is essentially the same argument about the connection of CVE to the torsional current as the direct evaluation of the Kubo formula in (20) with nontrivial e^0 . Combining the results (i) and (ii), we obtain the Kubo formula result (20). On the other hand, if we take the related but purely metric variation $u_\mu = h_{\mu\nu}$ to linear order, it is impossible to maintain independently $\delta\hat{\omega}_\mu \neq 0$ and $\delta e_\mu^a = 0$ ($\delta u_a \neq 0$) or vice versa in contrast to (i) and (ii). Note that it is possible to consider the transformation to the rotating frame via the nonhomogenous coordinate shift $\delta\Gamma_{\mu\rho}^\lambda$ [57], but this canceled in $\delta\omega_{\mu b}^a$ up to local Lorentz

rotations. Similarly, from the identification of momentum flux with T_i^0 or T^{ii} , the momentum cannot be independently sourced with e_i^a and u_i . Only the latter variation $\delta g_{0i} = \delta u_i$ was exclusively used in Ref. [80] [see Eqs. (2), (10), and (8)]; rather, this variation conjugate to momenta T^{ii} (8) with zero torsion produces the CVE.

This completes the detailed comparison of the Kubo formulas and chiral responses with and without torsion. To conclude, in our view, some of the conclusions of Ref. [80] (see also Ref. [38]) regarding torsion are not correct. In particular, torsion leads to nonvanishing, independent currents with similar anomaly coefficients as for CVE. We have obtained the response in terms of explicit torsional currents, equivalent to results of previous works, utilizing the same linear response Kubo formulas. The only case when torsion does not contribute is when it vanishes for the background (as is the case for the original Kubo formula computations). Again, it is a separate problem to ascribe the hydrodynamic degrees of freedom and constitutive relations with sources for a particular problem of chiral fermions and transport. Our results simply say that when torsion (i.e., tetrad and connection) is a hydrodynamic source it activates chiral currents at finite chemical potential and temperature in relativistic systems as the CTE. If only the metric enters, the response in terms of the CVE follows.

V. TWO GRAVITATIONAL ANOMALIES IN CHIRAL WEYL CONDENSATES

Let us now discuss CVE and CTE in terms of a system where we believe both anomalies can be sourced independently: nonrelativistic, chiral $p + ip$ condensate with Weyl quasiparticles at gap nodes. We note that, while superficially both anomaly terms originate from either normal component or condensate vorticity, the arguments we presented above and below are independent of this example. For more discussion, see Sec. VI.

A. Relativistic low-energy theory

We first present the low-energy theory for the normal component Bogoliubov fermions and the superfluid background; see, e.g., Ref. [18]. In the rotating vessel, the normal component undergoes solid body rotation with $\mathbf{v}_n = \boldsymbol{\Omega} \times \mathbf{r}$, which in equilibrium is canceled by a vortex lattice with spatially averaged $\langle\langle \mathbf{v}_s \rangle\rangle = \mathbf{v}_n$ over the unit cell.

The superfluid free energy transforms as [18]

$$f \rightarrow f[\boldsymbol{\rho}_s, \mathbf{l}, \mathbf{v}_s] + \mathbf{g}_s \cdot \mathbf{v} \quad (28)$$

under small Galilean transformations $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{v}t$. Here, the chiral $p + ip$ -wave gap amplitude is $\boldsymbol{\Delta} = \Delta_0(\hat{\mathbf{m}} + i\hat{\mathbf{n}})$, and $\hat{\mathbf{l}} = \hat{\mathbf{m}} \times \hat{\mathbf{n}}$ is a hydrodynamic Goldstone variable in addition to $\boldsymbol{\rho}_s, \mathbf{v}_s$. The total mass current $\mathbf{g} = \mathbf{g}_s + \mathbf{g}_n$ transforms as $\mathbf{g} \rightarrow \mathbf{g} + \rho\mathbf{v}$, leading to normal density $\rho_n = \rho\mathbf{1} - \rho_s$. Because of the gap nodes, this is density nonzero to first

order even at $T = 0$ in the presence of gap texture [18]. The mean-field quasiparticle equation of motion is $[i\partial_t - \mathcal{H}_{\text{BdG}}(\mathbf{k})]\psi$ with the Bogoliubov–de Gennes (BdG) Hamiltonian

$$\mathcal{H}_{\text{BdG}}(\mathbf{k}) = \begin{pmatrix} \epsilon(\mathbf{k}) & \frac{1}{2}\{\mathbf{k}, \mathbf{\Delta}\} \\ \frac{1}{2}\{\mathbf{k}, \mathbf{\Delta}^*\} & -\epsilon(-\mathbf{k}) \end{pmatrix} \quad (29)$$

where $\epsilon(\mathbf{k})$ is the normal state dispersion counted from the Fermi k_F and the brackets $\{a, b\} = ab + ba$ preserve Hermiticity with (weakly) coordinate dependent parameters. We assume equal spin-pairing and suppress spin indices. At the nodes $E_{\pm k_F \hat{l}} = 0$ are two Majorana-Weyl nodes (with twofold spin degeneracy). The quasiparticles are momentum eigenstates transforming under Galilean boost as

$$E_k(\mathbf{v}) = E_k + \mathbf{k} \cdot \mathbf{v}. \quad (30)$$

More generally, the normal state fermions transform as $\Psi(\mathbf{x}, t) \rightarrow e^{im\mathbf{v}\cdot\mathbf{x}}\Psi(\mathbf{x} + \mathbf{v}t, t)$, to lowest order in v , so the Bogoliubov fermions transform as $\psi(\mathbf{x}, t) \rightarrow e^{im(\mathbf{v}\cdot\mathbf{x} - \mu t)}\psi(\mathbf{x} + \mathbf{v}t, t)$. We generalize this to slowly varying $\mathbf{\Delta}(\mathbf{x}, t)$ and local transformations μ, \mathbf{v} in the gradient expansion, leading to the linearized equation of motion $i\partial_t - \mathcal{H}_{\text{BdG}}$ close to $k_F \hat{l}$ as

$$\tau^a e_a^\mu D_\mu \psi = \tau^a e_a^\mu [\partial_\mu - i\hat{\omega}_\mu]\psi = 0, \quad (31)$$

where τ^a are Pauli matrices in Nambu particle-hole space and we redefined $\psi \rightarrow e^{-1/2}\psi$ with $\hat{\omega}_\mu = \frac{1}{2}\omega_\mu^{ab}\tau_{ab} = \omega_\mu^{12}[\tau_1, \tau_2]/2 = m(-\mu, \mathbf{v})\tau^3$. The (torsionful) BdG spin connection D_a is a Galilean boost connection with respect to the Newtonian time and the combined $p + ip$ gauge symmetry $U(1)_{L_3-N/2}$ along \hat{l} [69], which for the BdG quasiparticles acquires the form of a relativistic spin-1/2 connection in the $a = 1, 2$ plane. We note that, using the anisotropic Newton-Cartan data [71], it can be equivalently written as the sum of a mass-current gauge field (30) and a \hat{l} -orthogonal Christoffel spin connection (which can be singular in the presence of vortices). The boost, spin connection, or Newton-Cartan structure is not needed here apart from the natural emergence of nonzero torsion along the \hat{l} direction [69–71].

In summary, the (inverse) tetrads from (29) and (31) are the linear expansion coefficients

$$\begin{aligned} e_0^\mu &= (1, -\mathbf{v}), & e_1^\mu &= (0, c_\perp \hat{\mathbf{m}}), \\ e_2^\mu &= (0, c_\perp \hat{\mathbf{n}}), & e_3^\mu &= (0, c_\parallel \hat{\mathbf{l}}), \end{aligned} \quad (32)$$

where $c_\perp = \Delta_0/k_F$ and $c_\parallel = v_F$. Inverting this, we obtain

$$\begin{aligned} e_\mu^0 &= (1, 0); & e_\mu^1 &= c_\perp (\hat{\mathbf{m}} \cdot \mathbf{v}, \hat{\mathbf{m}}), \\ e_\mu^2 &= c_\perp (\hat{\mathbf{n}} \cdot \mathbf{v}, \hat{\mathbf{n}}), & e_\mu^3 &= c_\parallel (\hat{\mathbf{l}} \cdot \mathbf{v}, \hat{\mathbf{l}}). \end{aligned} \quad (33)$$

There is the preferred (superfluid) frame where the normal component quasiparticles are at rest $\mathbf{v}_n = 0$, i.e., $\mathbf{v} = -\mathbf{v}_s$. The effective metric is secondary and is determined by the shape of the linear Weyl dispersion (31) via the superfluid background e_a^μ .

B. Cancellation of vortical and torsional anomaly currents

In rotating chiral Weyl superfluid/superconductor (e.g., $^3\text{He-A}$) with two components, normal and superfluid,

$$\mathbf{v}_n = \mathbf{u} = \mathbf{\Omega} \times \mathbf{r}, \quad \mathbf{\Omega} = \frac{1}{2}\boldsymbol{\omega}. \quad (34)$$

This produces the normal component CVE current $\mathbf{J}_5 \propto \mathbf{\Omega}$ from the CVE [114], via the Green's function identity [41,57]

$$G_{v_n}(\mathbf{x}, \mathbf{x}', k_0) = e^{-\frac{i}{2}\Omega_3 \Sigma_3 \partial_{k_0}} G_{v_n=0}(\mathbf{x}, \mathbf{x}', k_0), \quad (35)$$

where k_0 is the frequency, Σ_3 is the spin-1/2 rotation matrix along \hat{l} , and we have specialized to $\mathbf{\Omega} = \Omega_3 \hat{l}$ so that the system is effectively relativistic to linear order in \mathbf{k} .

On the other hand, from the BdG Hamiltonian to the linear order in \mathbf{v}_s , following Ref. [69], the geometry (32) induced by the superfluid background is torsionful. We set $\mu = \mu_s = 0$, $\mathbf{v} = -\mathbf{v}_s$ with $\partial_\mu \hat{l} = \partial_\mu \hat{\mathbf{m}} = \partial_\mu \hat{\mathbf{n}} = 0$ and obtain the spatial torsion

$$\begin{aligned} (\star e^a \wedge T_a)^k &= \frac{1}{c_\perp^2} \epsilon^{kij} e_i^m \partial_j e_t^m + O(v_s^2) \\ &= -\frac{1}{c_\perp^2} \epsilon^{kij} \partial_i v_{s_j}, \quad m = 1, 2, \end{aligned} \quad (36)$$

where the terms $\hat{l} \cdot \mathbf{v}_s = 0$ for the assumed vortex lattice configuration; see the discussion below in Sec. VI. Note that this term is subleading in gradients compared to Ref. [69] but needs to be here included, since $\nabla \times \mathbf{v}_n = \mathbf{\Omega}$ is nonzero.

The combination of two gravitational anomalies gives CTE + CVE:

$$\begin{aligned} J_5^\mu &= \frac{T^2}{12} \epsilon^{\mu\nu\lambda\rho} \left[\frac{1}{2} \eta_{ab} e_\nu^a T_{\lambda\rho}^b + u_\nu \partial_\lambda u_\rho \right] \\ &= \frac{T^2}{12c_\perp^2} (\nabla \times \mathbf{v}_s - 2\mathbf{\Omega}). \end{aligned} \quad (37)$$

In the equilibrium state in the rotating cryostat, the vortex lattice has spatially averaged vorticity $\langle\langle \nabla \times \mathbf{v}_s \rangle\rangle = 2\mathbf{\Omega}$, which corresponds to $\langle\langle \mathbf{v}_s \rangle\rangle = \mathbf{v}_n = \mathbf{\Omega} \times \mathbf{r}$ in equilibrium.

That is why in thermodynamic equilibrium the two anomalies cancel each other, $\langle\langle \mathbf{J}_5 \rangle\rangle = 0$. The local currents exist in the vortex lattice, $\nabla \times \mathbf{v}_s - \langle\langle \nabla \times \mathbf{v}_s \rangle\rangle \neq 0$. But the total current along the rotation axis is zero.

This demonstrates that a thermal version of the Bloch theorem [115], i.e., the absence of particle current in equilibrium, is applicable to the chiral vortical current (see, e.g., Ref. [116] for the axial CVE current in $^3\text{He-A}$). Here, the Bloch theorem follows as the consequence of the anomaly current cancellation in a chiral condensate with rotating normal component chiral fermions.

VI. CONCLUSION

By extending the hydrodynamic chiral gravitational responses to nonzero torsion, we computed the linear response anomalous Kubo formula. We clarified the torsionful hydrodynamic variations $e_\mu^a, \omega_{\mu b}^a$ in comparison to metric variations $u_\mu, g_{\mu\nu}$ in anomalous chiral transport. We identify a simple reason why the CVE contributes at linear order with the gravitational T^2 term: it is directly related to the mixed torsional gravitational anomaly with coefficient $t_\chi = d_\chi$, and the two responses can be related by tuning the torsion part of the connection to be nonzero and varying the timelike $e_\mu^0 \sim u_\mu$. This new viewpoint complements the earlier arguments for CVE with thermodynamic variations and gravitational background sources, formulated in terms of u_μ and $g_{\mu\nu}$ and the mixed gravitational R^2 anomaly without torsion.

In particular, recent work on (non)relativistic hydrodynamic transport with torsion [54,81–83] did not take into account the presence of the Nieh-Yan anomaly terms. See, however, the more recent works [70,83,84] focusing on finite, relativistic UV anomaly terms which we did not discuss here. The results we find with the relativistic Kubo formulas are in exact agreement with previous results, including the simple but physically compelling Landau level approach with and without UV cutoffs [71,76].

The relation of the CTE and CVE is reminiscent of the gravitational anomalies in general; for example, it is known that the nonconservation of the energy-momentum tensor is not independent of the Lorentz anomaly (i.e., antisymmetric T^{ab}) and that the anomaly can be transferred from one form to the other, as well as written in several equivalent forms [6]. The choice where to place the anomaly should be made in terms of the independent degrees of freedom, the Lorentz and torsional anomalies being just manifestations of the presence of degrees of freedom independent from the metric. For hydrodynamic transport, the essential difference is whether the frame fields and connection enter the constitutive relations with sources independently from the metric. In the limit $k \rightarrow 0$ at zero frequency, both conductivities are equal and constitute two parts: a chemical potential term μ^2 induced by the chiral gauge anomaly and a T^2 term from the gravitational anomaly. In retrospect, this

fits perfectly into the different ways of understanding torsion in terms of a momentum-dependent chiral gauge field corresponding to totally antisymmetric torsion. Finally, to discern between the two anomalies, one can probe the finite frequency and momentum conductivities (20), which are different for the CTE and CVE, since the two independent contributions coincide only in the limit $k \rightarrow 0$ at zero frequency. On the other hand, it is known that the “anomaly quantization” of the CVE, and therefore also the CTE, is only valid to the lowest nontrivial order $O(k^2)$ and that, e.g., dynamical gauge fields contribute to the conductivity [9,50].

We discussed the two anomalies for nonrelativistic chiral Weyl superfluids and superconductors, which in the low-energy approximation have chiral (Majorana-)Weyl fermions. The condensate background fields can be modeled as relativistic fermions on Riemann-Cartan or anisotropic Newton-Cartan geometries [69,71]. Along the anisotropy direction and linear approximation, the geometry is effectively relativistic, and the relativistic CVE and CTE results can be applied. The CVE is induced by the rotating normal component [41], whereas the torsionful tetrad arises from the superfluid component with \mathbf{v}_s . While CTE here superficially originates from condensate vorticity, we note that, by so-called Mermin-Ho relations, \mathbf{v}_s is related in hydrodynamics to the order parameter frame fields $e_i^a = \{\hat{\mathbf{m}}, \hat{\mathbf{n}}, \hat{\mathbf{l}}\}$, of which only the $\hat{\mathbf{l}}$ is a well-posed hydrodynamic variable in addition to $\mu, T, \rho_s, \rho_n, \mathbf{v}_s, \mathbf{v}_n$ (spin is neglected). The perpendicular $\hat{\mathbf{m}}, \hat{\mathbf{n}}$ -components contribute to \mathbf{v}_s and represent gauge degrees of freedom in terms of the combined orbital-phase symmetry of the superfluid (superconductor) [18]. In the absence of relativistic systems with torsion, this comparison of the two anomalies with independent experimentally available sources is the best we can do. To see the relation of the two anomalies, we had to “bias” \mathbf{v}_s with \mathbf{v}_n and go higher order in gradients as compared to Ref. [69]. In this case, the absence of chiral currents in equilibrium is obtained by the cancellation of the two anomaly currents, spatially averaged over (the unit cell of) the vortex lattice. To maintain the validity of the relativistic approximation with spin-1/2 chiral fermions, we considered the case where $\hat{\mathbf{l}}$ is constant and along rotation axis Ω , corresponding to pure phase vortices [117].

In real rotating chiral superfluid $^3\text{He-A}$, the vortex lattice is created by quenching the critical rotation velocity whereby many different vortex lattices form quasiequilibrium thermal states, with barriers exceeding 5 orders of magnitude compared to T . It is therefore expected that the anomaly can be “transferred” between the $\hat{\mathbf{l}}$ vector, \mathbf{v}_s and \mathbf{v}_n components of the superfluid, and this choice is neither boundary condition nor gauge independent (in terms of the gap parameters). We did not, however, invoke anomaly inflow at vortex cores or at the boundary in the cancellation of the bulk anomalous current. It would be interesting to further study this nonrelativistic problem from the

geometrical perspective, especially in terms of the vortex structures [118]. There is no particular hydrodynamical reason to restrict to low angular velocities, temperatures (compared to T_c) or chemical potentials. Relativistically, to higher orders [41,52,57,61,113],

$$\mathbf{J}_5 = \left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12} + \frac{\Omega^2}{48\pi^2} - \frac{m^2}{8\pi^2} - \frac{R}{96\pi^2} \right) \mathbf{\Omega} + \dots \quad (38)$$

with m the fermion mass and R the scalar curvature in the plane perpendicular to $\mathbf{\Omega}$. Even better, the higher-order terms should be worked out in detail in terms of the full nonrelativistic geometry and compared to the low-energy (relativistic) results and the regime of validity of the chiral transport with Weyl fermions, including torsion [77]. The ensuing effective hydrodynamic actions for the Goldstones should be worked out in detail.

On this note, for conceptual (and calculational [41,57]) clarity, we interpreted here the CVE exclusively from the rotation $\mathbf{\Omega}$ in the preferred frame with torsion. For consistency of the background, the CTE and CVE should be possible to understand solely in terms of the tetrad/metric, where the both CVE source \mathbf{v}_n and (torsionful) \mathbf{v}_s tetrad directly contribute. In case of torsion, metric variations are superseded by those corresponding to the tetrad and connection. Moreover, Galilean invariance immediately suggests that $\mathbf{v}_s - \mathbf{v}_n$ should enter as the nontrivial velocity. Moreover, the nonrelativistic extension of CTE could explain the $T = 0$ anomalous angular momentum terms in the chiral $p + ip$ system [114]. Other nonrelativistic systems where torsion is directly relevant are Weyl semimetals with dislocations; in the continuum approximation, the dislocation density arises from the torsion of globally nontrivial tetrad variation $\delta e_\mu^a = \partial_\mu u^a$, where u^a is now the elastic displacement. In the case where u^a is globally continuous, the nontrivial field strength needed for tetrads (or strain pseudo-gauge-fields) vanishes. In fact, torsion $T_{\mu\nu}^a$ is equivalent to ‘‘spatial vorticity’’ of the u^a from dislocations, and the findings of this paper can be applied with minimum alterations to axially twisted semimetals. What is missing is the spatial analogy to the static, equilibrium CVE, and CTE is expected to play its role. In addition to twisting, the corresponding bias for dislocations could be the *real* geometry/gravitational field, in the absence of rotation [119]. Nevertheless, see Ref. [37] for some recent considerations of the anomalous transport and CME *without* dislocations (and therefore no geometric torsion).

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APPENDIX A: CURVATURE CONVENTIONS WITH TORSION

We discuss tetrads e_μ^a and connections $\hat{\Gamma}_\mu^\lambda, \hat{\omega}_\mu$ that are metric $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ compatible, $\nabla_\lambda g_{\mu\nu} = \nabla_\lambda \eta_{ab} = 0$,

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\lambda e_\lambda^a + \omega_{\mu b}^a e_\nu^b = 0, \quad (A1)$$

$$\Rightarrow \omega_{\mu b}^a = e_\lambda^a \Gamma_{\mu\nu}^\lambda e_\nu^b + e_\nu^a \partial_\mu e_\nu^a. \quad (A2)$$

We can form the well-defined tensor 1-forms in the tangent space $e^a = e_\mu^a dx^\mu$ and $\omega^a_b = \omega_{\mu b}^a dx^\mu$ that transform under local Lorentz rotations, the latter as a connection. The tetrad and connection are *a priori* independent quantities with field strength tensors that transform homogeneously,

$$T^a = de^a + \omega_{\mu\nu}^a \wedge e^\nu = \frac{1}{2} T_{\mu\nu}^a dx^\mu \wedge dx^\nu \quad (A3)$$

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = \frac{1}{2} R_{\mu\nu b}^a dx^\mu \wedge dx^\nu, \quad (A4)$$

where in the coordinate basis

$$T_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda \quad (A5)$$

$$R_{\mu\nu\rho}^\lambda = \partial_\mu \Gamma_{\nu\rho}^\lambda + \Gamma_{\mu\tau}^\lambda \Gamma_{\nu\rho}^\tau - (\mu \leftrightarrow \nu). \quad (A6)$$

The curvature takes the usual form but depends on torsion through $\Gamma_{\mu\nu}^\lambda$. Without loss of generality, we can set

$$\Gamma_{\mu\nu}^\lambda = \overset{\circ}{\Gamma}_{\mu\nu}^\lambda + K_{\mu\nu}^\lambda, \quad (A7)$$

where $K_{\mu\nu}^\lambda = \frac{1}{2}(T_{\mu\nu}^\lambda + T_{\nu\mu}^\lambda - T_{\mu\nu}^\lambda)$ is the contorsion and $\overset{\circ}{\Gamma}_{\mu\nu}^\lambda[g_{\mu\nu}]$ is the symmetric Christoffel connection fully determined by the metric (or, equivalently, the tetrad e_μ^a). The analogous formula holds for the metric compatible spin connection $\hat{\omega} = \overset{\circ}{\omega}[e^a] + \hat{\omega}_K$.

For torsionful spacetimes the two tensors T^a and R^a_b characterize the geometry, and we can regard e^a, ω^a_b as independent. Another option is to take e^a and (con)torsion T^a (K^a_b) as the independent variables. The corresponding variations are collected in (9).

APPENDIX B: TORSIONAL NIEH-YAN FORM AND ANOMALY

The Nieh-Yan form [87] is a locally exact purely torsional form, given in terms of the 3-form and corresponding 1-form dual,

$$e_a \wedge T^a = \frac{\eta_{ab}}{2} e_\mu^a T_{\nu\lambda}^b dx^\mu \wedge dx^\nu \wedge dx^\lambda, \quad (B1)$$

$$\star e_a \wedge T^a = \frac{\eta_{ab}}{2} \epsilon^{\mu\nu\lambda} e_\mu^a T_{\nu\lambda}^b dx^\rho, \quad (B2)$$

which are both (locally) Lorentz invariant tensors and therefore well defined, in contrast to standard Chern-Simons anomaly currents. The Bianchi identities with torsion imply that

$$d(e_a \wedge T^a) = T^a \wedge T_a - e^a \wedge e^b \wedge R_{ab}. \quad (\text{B3})$$

This is an independent closed curvature-invariant form that vanishes if and only if torsion is zero. The Nieh-Yan anomaly equation is

$$\tilde{\nabla}_\mu J_5^\mu = \frac{\Lambda^2}{4\pi^2} \star d(e^a \wedge T_a), \quad (\text{B4})$$

where $\tilde{\nabla}_\mu = \nabla_\mu - T_{\lambda\mu}^\lambda$ is the covariant divergence with torsion and Λ is a dimensionful UV scale with canonical dimensions of momentum. See, e.g., Ref. [92] for a discussion on the topological properties of the NY form and torsional chiral anomaly. For relativistic fermions, the coefficient Λ is interpreted as a UV cutoff that diverges for Weyl or Dirac fermions in vacuum. Nevertheless, and in stark contrast to traditional anomalies, this problematic cutoff divergence can be canceled by the local counterterm

$$S_{c.t.}[A_5, e^a, \hat{\omega}] = -\frac{\Lambda^2}{4\pi^2} \int A_5 \wedge e_a \wedge T^a \quad (\text{B5})$$

in the effective action, which breaks no additional symmetries, i.e., neither vector U(1) nor Lorentz or diffeomorphism symmetries. In contrast, for more complicated relativistic or condensed matter systems, the fate of the UV anomaly is determined by the UV completion of the low-energy chiral theory. See, e.g., Refs. [80,83,84,91–94] and [62–64,69–71] for more discussions on the UV anomaly in relativistic and nonrelativistic systems. For example, the NY anomaly term can be generated in hydrodynamics with matter, as we discuss in the main text, where Λ is supplemented by the available (low-energy) scales T , μ , m , etc., which we focus on in the main text.

APPENDIX C: METRIC VELOCITY PERTURBATION

Around flat space, we perturb $g_{\mu\nu}$ as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In the applications to CVE, $h_{ii} = u_i$, with all other components zero to linear order. The variation of the tetrad is in (7). The Christoffel connection $\overset{\circ}{\Gamma}_{\lambda\mu\nu} = g_{\lambda\rho} \overset{\circ}{\Gamma}_{\mu\nu}^\rho$ changes as

$$\delta \overset{\circ}{\Gamma}_{\lambda\mu\nu} = \frac{1}{2} (\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}) \quad (\text{C1})$$

$$\delta \overset{\circ}{\Gamma}_{ij} = \frac{1}{2} (\partial_i h_{ij} + \partial_j h_{ii}) \quad (\text{C2})$$

$$\delta \overset{\circ}{\Gamma}_{itj} = -\frac{1}{2} (\partial_i h_{tj} - \partial_j h_{ti}) \quad (\text{C3})$$

$$\delta \overset{\circ}{\Gamma}_{itt} = \partial_t h_{ii}. \quad (\text{C4})$$

The spin connection is

$$\delta \overset{\circ}{\omega}_{ab\mu} = e_{a\lambda} \delta \overset{\circ}{\Gamma}_{\mu\nu}^{\lambda} e_b^\nu + e_{a\nu} \partial_\mu e_b^\nu \quad (\text{C5})$$

$$= \delta \overset{\circ}{\Gamma}_{ab\mu} + e_{a\nu} \partial_\mu e_b^\nu, \quad (\text{C6})$$

where $e_{a\nu} \partial_\mu e_b^\nu = \delta_{a0} \partial_\mu u_m \delta_{bm}$. We get

$$\overset{\circ}{\omega}_{0im} = \partial_t u_m, \quad (\text{C7})$$

$$\overset{\circ}{\omega}_{0im} = \frac{1}{2} (\partial_i u_m - \partial_m u_i), \quad (\text{C8})$$

$$\overset{\circ}{\omega}_{mtn} = \frac{1}{2} (\partial_m u_n - \partial_n u_m). \quad (\text{C9})$$

These lead to $T_{\mu\nu}^a = 0 + O(u^2)$ as expected. In the absence of torsion, the linear response to $u_i = h_{0i}$ is to be computed with the CVE Kubo formula of $\delta g_{\mu\nu}$.

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