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Matched waves and unexpected resonances: variety of boundary conditions

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Abstract

This article discusses certain interesting findings on phenomena that take place when electromagnetic fields interact with a surface on which given boundary conditions are enforced. In particular, the concept of matched waves is analyzed and illustrated. Furthermore, observations are made on resonance spectra that may accompany subwavelength scatterers with isotropic impedance boundary condition. This invited article is based on the Commission B (Fields and Waves) keynote presentation of the 2019 URSI Asia–Pacific Radio Science Conference held in New Delhi, India, on 9–15 March 2019.

1 Introduction

To solve electromagnetic problems, in other words electric and magnetic fields in a given domain, fields need to be found that satisfy Maxwell equations. The freedom of solutions that the equations allow is restricted by boundary conditions that the fields need to fulfill on the boundary of the domain of interest. Boundary conditions are indeed essential in electromagnetics and they are in the focus of this article.

In electromagnetics, a related concept to a boundary is an interface. The interface is a two-dimensional surface which separates two homogeneous media. On an interface, continuity conditions between the tangential components of the electric and magnetic fields are enforced, and likewise the continuity of the normal components of the flux densities is required [1]. However, despite the continuity of certain components over the interface, the total vectors are changed in general in each of the four cases, the magnitude of the change being determined by the contrasts of the permittivity and permeability between the two media.

Hence for the interface problem, fields exist on both sides of the surface. This marks a qualitative difference to the boundary problem where the fields are only in the domain to be considered. The space beyond this boundary is irrelevant: fields have no meaning there. Another way to look at the situation is to replace the boundary problem by an equivalent free-space problem where sources (electric and magnetic charge and current densities) exist on that two-dimensional surface that was the boundary in the original problem. These sources (together with the primary sources of the original problem) create the correct fields in the original domain, but they radiate zero fields into the domain on the non-interesting side of the boundary.

In classical electromagnetics, much-used boundary conditions are the PEC (perfect electric conductor) and PMC (perfect magnetic conductor) boundaries, on which the tangential components of the electric or tangential components of the magnetic field vector has to vanish, respectively. Such conditions are rather powerful in their "short-circuiting" capability, which leads to the fact that electromagnetic waves hitting such boundaries will be very strong reflected. Another boundary condition which is used to approximate highly conducting structures is the so-called impedance-boundary condition which was introduced in the 1940's [2, 3] with the objective to facilitate the analysis of radio-wave propagation over ground or sea. Towards the end of the century, different ways of approximating layered and other type of structures with effective boundary conditions were developed and documented [4, 5, 6].

However, due to the vectorial character of the electric and magnetic fields, the variety of different possibilities of boundary conditions is very large. Let us take a look on their diversity in the following.

2 Classification of boundary conditions

Boundary conditions are given as relations of the tangential and/or normal components of the electric and magnetic fields ($\mathbf{E}, \mathbf{H}$) or flux densites ($\mathbf{D}, \mathbf{B}$) at the boundary under consideration. The unit normal vector $\mathbf{n}$ of the boundary (see Figure 1) is used to project these components (the dot product of $\mathbf{n}$ with the field vector gives the normal component, and the cross product returns the (90°) rotated tangential component).

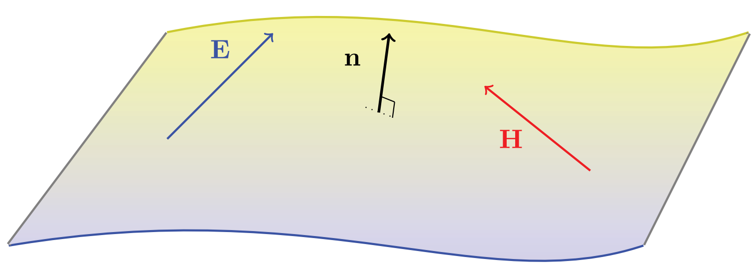

Figure 1: Boundary surface represented by a two-dimensional sheet with unit normal vector $\mathbf{n}$. {fig:bc}

The classification of boundary conditions can be done using different principles. An obvious way to put the conditions into an order from simple ones to more complex conditions by the number of degrees of freedom that they contain. Another possibility, to be followed here, is to look at classes with similar character, without consideration of the number of free parameters of a given boundary condition. Nevertheless, in the following we start from the well-known ones.

PEC, PMB, and PEMC boundaries

As already mentioned above, the perfect electric conductor boundary forces the tangential part of the electric field to vanish:

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{(PEC)} \tag{1}$$ {PEC}

while the perfect magnetic conductor boundary is its dual

$$\mathbf{n} \times \mathbf{H} = 0 \quad \text{(PMC)} \tag{2}$$ {PMC}

A generalization of these two boundary conditions is *perfect electromagnetic conductor* (PEMC) boundary, which appears at the boundary of the PEMC medium. This medium was introduced in 2005 [7], defined by the following fundamental constitutive relations:

$$\mathbf{D} = M\mathbf{B}, \quad \mathbf{H} = -M\mathbf{E} \tag{3}$$ {PEMC1}

where M is the PEMC parameter, having an admittance character. Hence the PEMC boundary condition reads

$$\mathbf{n} \times (\mathbf{H} + M\mathbf{E}) = 0 \quad \text{(PEMC)} \tag{4}$$ {PEMC}

which form immediately shows that PEC and PMC are special cases ($1/M = 0$) and PMC ($M = 0$) of the PEMC boundary condition.[1]

Impedance boundaries

The so-called impedance boundary condition (IBC) requires that the tangential electric and magnetic fields stand in a given relation. Due to the fact that the tangential components of the fields have a two-dimensional interconnection, the space of the impedance boundary condition has to be described by four scalars. Formally, the impedance condition is written

$$\mathbf{E}_t = \overline{\overline{\mathsf{Z}}}_s \cdot (\mathbf{n} \times \mathbf{H}_t) \tag{5}$$ {ibc1}

between the tangential electric and magnetic fields at the boundary

$$\mathbf{E}_t = -\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \mathbf{n}\mathbf{n}{}^{\times}_{\times}\mathbf{E}, \quad \mathbf{H}_t = -\mathbf{n} \times (\mathbf{n} \times \mathbf{H})) = \mathbf{n}\mathbf{n}{}^{\times}_{\times}\mathbf{H} \tag{6}$$ {key}

[1] In the four-dimensional electromagnetics formalism, the PEMC condition has been shown to have a natural connection to axion electrodynamics [8, 9, 10].

Here the surface impedance dyadic $\overline{\overline{\mathsf{Z}}}_s$ is two-dimensional, in the plane perpendicular to $\mathbf{n}$, in other words $\mathbf{n} \cdot \overline{\overline{\mathsf{Z}}}_s = 0$ and $\overline{\overline{\mathsf{Z}}}_s \cdot \mathbf{n} = 0$.

The IBC condition (5) gives, as special cases, PEC boundary condition ($\overline{\overline{\mathsf{Z}}}_s = 0$), and the PMC boundary condition ($\overline{\overline{\mathsf{Z}}}_s{}^{-1} = 0$).

A two-dimensional dyadic can be decomposed into a physically interpretable manner using the following four elementary base dyadics [11]:

$$\begin{aligned} \overline{\overline{\mathsf{I}}}_t &= \mathbf{uu} + \mathbf{vv} \\ \overline{\overline{\mathsf{J}}} &= \mathbf{n} \times \overline{\overline{\mathsf{I}}}_t = \mathbf{vu} - \mathbf{uv} \\ \overline{\overline{\mathsf{K}}} &= \mathbf{uu} - \mathbf{vv} \\ \overline{\overline{\mathsf{L}}} &= \mathbf{uv} + \mathbf{vu} \end{aligned} \tag{7}$$

where the unit vectors $\mathbf{u}$ and $\mathbf{v}$ span the two-dimensional plane under consideration and $\mathbf{u}, \mathbf{v}, \mathbf{n}$ form an orthonormal set.

The isotropic IBC arises from $\overline{\overline{\mathsf{Z}}}_s = Z_s \overline{\overline{\mathsf{I}}}_t$ where Z_s is a complex scalar. For such an impedance surface to be lossless, Z_s has to be purely imaginary [11, Section 3.6]. If the real part of Z_s is positive, the surface is dissipative, and correspondingly the negative real part means an active surface.

The case $\overline{\overline{\mathsf{Z}}}_s = Z_s \overline{\overline{\mathsf{J}}}$ leads to the PEMC surface (Equation (4)), with the connection $M Z_s = 1$ to the PEMC admittance parameter. For a PEMC boundary to be lossless, the Z_s (or M) parameter has to be real.

In contrast to $\overline{\overline{\mathsf{I}}}_t$ and $\overline{\overline{\mathsf{J}}}$, the two remaining base dyadics $\overline{\overline{\mathsf{K}}}$ and $\overline{\overline{\mathsf{L}}}$ are anisotropic. Hence any surface impedance dyadic with non-zero component of $\overline{\overline{\mathsf{K}}}$ or $\overline{\overline{\mathsf{L}}}$ has a dependence on the direction of the fields in the plane of the surface. The losslessness requirement for such surfaces is that Z_s is purely imaginary in the expansions $\overline{\overline{\mathsf{Z}}}_s = Z_s \overline{\overline{\mathsf{K}}}$ or $\overline{\overline{\mathsf{Z}}}_s = Z_s \overline{\overline{\mathsf{L}}}$.

Reciprocity is an important property in electromagnetics [12] and also for boundary conditions. The mathematical requirement that an impedance boundary be reciprocal is that the dyadic $\overline{\overline{\mathsf{Z}}}_s$ is symmetric [11]. Therefore $\overline{\overline{\mathsf{I}}}_t, \overline{\overline{\mathsf{K}}}$, and $\overline{\overline{\mathsf{L}}}$ are allowed components in the dyadic $\overline{\overline{\mathsf{Z}}}_s$ for a reciprocal IBC. Indeed, the PEMC boundary is non-reciprocal, since $(1/M)\mathbf{n} \times \overline{\overline{\mathsf{I}}}_t$ is not symmetric.

Boundary conditions with normal components of fields

While the impedance boundary condition linked the *tangential* components of the electric and magnetic fields with each other, a complementary approach is to enforce a condition on their *normal* components. Such a non-traditional viewpoint has been shown to lead into interesting conceptual interpretations of phenomena appearing in the wave interaction with such a boundary (like matched waves, to be discussed more closely later in this article), as well as promising application perspectives for scattering by objects with this type of boundary (such as zero-backscattering objects [13]).

Probably the simplest of such boundary conditions is one in which the normal components of the electric and magnetic flux densities vanish:

$$\mathbf{n} \cdot \mathbf{D} = 0, \quad \mathbf{n} \cdot \mathbf{B} = 0 \tag{8}$$ {DB}

Originally introduced already in 1959 [14], this boundary condition has received attention in recent years under the label "DB boundary condition" [15, 16, 17]. For isotropic media (like free space in Figure 1), conditions (8) are tantamount to corresponding conditions for the fields:

$$\mathbf{n} \cdot \mathbf{E} = 0, \quad \mathbf{n} \cdot \mathbf{H} = 0 \tag{9}$$ {DB1}

A related boundary condition with focus on normal components of the displacements is D'B' condition where the normal derivatives of the normal components vanish. For a planar boundary, this means

$$(\mathbf{n} \cdot \nabla)(\mathbf{n} \cdot \mathbf{D}) = \partial_n D_n = 0, \quad (\mathbf{n} \cdot \nabla)(\mathbf{n} \cdot \mathbf{B}) = \partial_n B_n = 0 \tag{10}$$ {DpBp}

An interesting connection back to PEC and PMC boundaries comes from the "mixed boundaries": refering to the normal derivative with the prime, the D'B condition is equivalent to PEC, and the DB' condition to PMC [18].

It is essential to note the fundamental character of DB and D'B' boundary conditions: they do not contain any free parameter. In that respect (sharing with PEC and PMC the vanishing number of degrees of freedom) they are more primary than impedance boundary condition.

A further generalization from DB and D'B' is the so-called "mixed-impedance boundary" in which the conditions combine the normal component and its derivative, separately for the electric and magnetic flux densities [19].

Hybrid boundary conditions

Combining the tangential and normal components of the fields into the same boundary condition takes us deeper into the multidimensional space of the variety of boundary conditions. A very general representation of such hybrid boundary conditions is the following [20]:

$$\begin{aligned} \alpha_1 \mathbf{n} \cdot c\mathbf{B} + \beta_1 \mathbf{n} \cdot c\eta_0 \mathbf{D} + \mathbf{a}_{1t} \cdot \mathbf{E} + \mathbf{b}_{1t} \cdot \eta_0 \mathbf{H} &= 0 \\ \alpha_2 \mathbf{n} \cdot c\mathbf{B} + \beta_2 \mathbf{n} \cdot c\eta_0 \mathbf{D} + \mathbf{a}_{2t} \cdot \mathbf{E} + \mathbf{b}_{2t} \cdot \eta_0 \mathbf{H} &= 0 \end{aligned} \tag{11}$$ {GB}

where the normalization with constants $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ and $c = 1/\sqrt{\mu_0\epsilon_0}$ makes the parameters of the boundary condition $\alpha_i, \beta_i, \mathbf{a}_{it}, \mathbf{b}_{it}$ dimensionless. Here the vectors $\mathbf{a}_{it}$ and $\mathbf{b}_{it}$ are transversal: $\mathbf{n} \cdot \mathbf{a}_{it} = \mathbf{n} \cdot \mathbf{b}_{it} = 0$.

Due to the fact that the electric and magnetic fields and flux densities are connected, and that the equations (11) are homogeneous, there are only 8 free boundary parameters in these equations. Nevertheless, this corresponds a very large space of potential boundary conditions. As an example of hybrid boundary conditions that connect the tangential and normal components of the fields, we can mention the SHDB boundary [21]. This is a one-parameter boundary with the following conditions:

$$\mathbf{a}_t \cdot \mathbf{E} + \alpha\mathbf{n} \cdot \eta_0\mathbf{H} = 0, \quad \alpha\mathbf{n} \cdot \mathbf{E} - \mathbf{a}_t \cdot \eta_0\mathbf{H} = 0 \tag{12}$$ {SHDBcond}

Obviously, the choice for the parameter $\alpha^{-1} = 0$ returns the DB boundary (Equation (9)). The other limit ($\alpha = 0$) gives the condition $\mathbf{a}_t \cdot \mathbf{E} = 0, \mathbf{a}_t \cdot \mathbf{H} = 0$ which is the so-called soft-and-hard surface (SH). This boundary condition, coined by Per-Simon Kildal [22], can be approximated by a corrudated conducting surface, and finds plenty of applications in microwave components and antenna technology.

Further examples of hybrid boundary conditions are the generalized SHDB surface in which the transversal vectors $\mathbf{a}_t$ in the two conditions in (12) can point in two different transversal directions [23], and the E boundary [20] for which the condition is $\mathbf{a}_1 \cdot \mathbf{E} = 0$ and $\mathbf{a}_2 \cdot \mathbf{E} = 0$, here $\mathbf{a}_1$ and $\mathbf{a}_2$ being arbitrary vectors that satisfy $\mathbf{a}_1 \times \mathbf{a}_2 \neq 0$. (When these two vectors are tangential to the boundary the E condition becomes equal to the special case of PEC.)

3 Matched waves

An interesting concept in connection to boundary conditions is that of a matched wave. The way to solve a problem of reflection of a plane wave from a boundary or interface is to find such a reflected wave such that the sum of the incident and reflected waves satisfies the boundary conditions. By a matched wave we mean an incident wave with such a polarization state that the boundary condition is already satisfied without any reflected wave. The matched wave is a single wave which can have phase velocity towards the surface ("incident" wave), or away from the surface ("reflected" wave). In the former case, the boundary is absorbing, and in the latter one it is active.

Let us treat the reflection problem of a plane wave from a general boundary. The incident $\mathbf{E}^i(\mathbf{r})$ and reflected $\mathbf{E}^r(\mathbf{r})$ electric field functions read

$$\mathbf{E}^i(\mathbf{r}) = \mathbf{E}^i\, \mathrm{e}^{-\mathrm{j}\mathbf{k}^i\cdot\mathbf{r}}, \quad \mathbf{E}^r(\mathbf{r}) = \mathbf{E}^r\, \mathrm{e}^{-\mathrm{j}\mathbf{k}^r\cdot\mathbf{r}} \tag{13}$$ {electricfields}

where we assume the time-harmonic dependence $\exp(\mathrm{j}\omega t)$. The magnetic fields come from Faraday law as

$$k_0\eta_0\mathbf{H}^i(\mathbf{r}) = \mathbf{k}^i \times \mathbf{E}^i(\mathbf{r}), \quad k_0\eta_0\mathbf{H}^r(\mathbf{r}) = \mathbf{k}^r \times \mathbf{E}^r(\mathbf{r}) \tag{14}$$ {magneticfields}

Figure 2 shows the geometry of the reflection problem. The incident and reflected wave vectors have the same transversal component $\mathbf{k}_t$ and opposite normal components k_n:

$$\mathbf{k}^i = \mathbf{k}_t - k_n\mathbf{n}, \quad \mathbf{k}^r = \mathbf{k}_t + k_n\mathbf{n} \tag{15}$$ {kikr}

and their magnitudes are the same:

$$\mathbf{k}^i \cdot \mathbf{k}^i = k_0^2 = \omega^2\mu_0\epsilon_0, \quad \mathbf{k}^r \cdot \mathbf{k}^r = k_0^2 = \omega^2\mu_0\epsilon_0 \tag{16}$$ {ko}

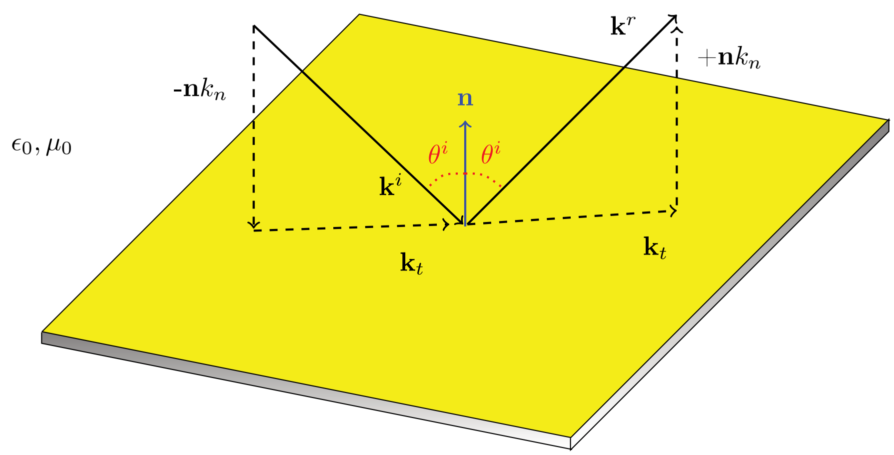

Figure 2: Incident and reflected wave vectors $\mathbf{k}^i, \mathbf{k}^r$. {fig:r}

Matched waves at DB boundary

For an illustrative view on matched waves, consider plane wave reflection from a DB boundary which forces the normal components of both fields to vanish (Equation (9)). The plane wave with wave vector $\mathbf{k}^i$ impinges in an oblique angle on the boundary, and the wave vector of the reflected wave $\mathbf{k}^r$ has the same transversal wave vector as $\mathbf{k}^i$.

In Figure 3, the polarization of the incident (and reflected) wave is perpendicular (TE polarization). Hence the electric field does not have any normal component, and the DB condition applies only on the magnetic field: the reflected wave needs to have a normal component that is opposite to that of the incident wave. The result is that the electric field changes sign in reflection: the reflection is equal to that from a PEC surface.

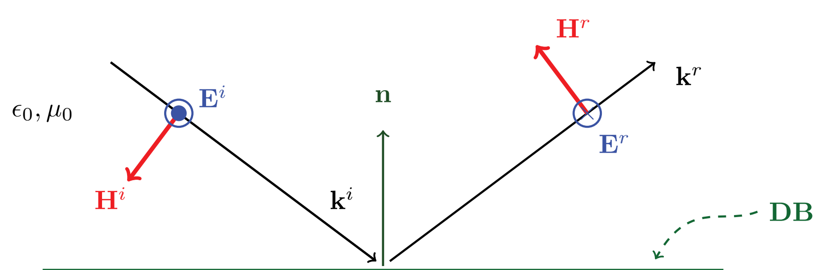

Figure 3: Reflection of a perpendicularly polarized plane wave from the DB boundary. The boundary condition only restricts the magnetic field, forcing its normal component to vanish. The reflection is equivalent to a reflection from a PEC plane. {fig:DB1}

On the other hand, the complementary phenomenon takes place for a parallel-polarized plane wave (TM polarization) reflecting from a DB boundary, illustrated in Figure 4. The DB condition requires that the normal components of the incident and reflected electric fields have to be equal in magnitude and opposite in direction. This leads to the situation that the magnetic field is short-circuited at the boundary, which is equivalent to a reflection from a PMC boundary.

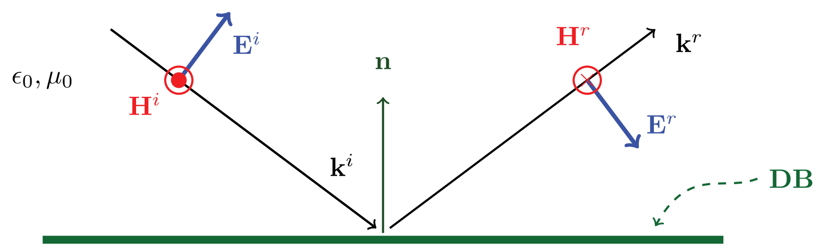

Figure 4: Like in Figure 3, for parallel polarization incidence. Now the boundary condition applies on the electric field, forcing its normal component to vanish. The reflection is the same as from a PMC boundary. {fig:DB2}

Therefore, the DB boundary acts as a PEC boundary for TE waves and as a PMC boundary for TM waves. The question arises what happens for a TEM wave, in other words a wave with normal incidence for which both the electric and magnetic fields are in the plane of the boundary and do not have any normal component?

The answer is that there is no reflected wave. The DB boundary condition is already satisfied by the incident wave. A normally incident wave is a wave matched to the DB boundary.

Matched waves at a general boundary

Mathced waves can be found for all types of boundary conditions. It may happen that some matched waves are inhomogeneous plane waves, like surface or leaky waves but still there exists a systematic wave to determine conditions for wave vectors that correspond to matched waves for any arbitrary boundary condition [20].

The general hybrid boundary condition (11) can be written in the form

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{E} + \mathbf{b}_1 \cdot \eta_0 \mathbf{H} &= 0 \\ \mathbf{a}_2 \cdot \mathbf{E} + \mathbf{b}_2 \cdot \eta_0 \mathbf{H} &= 0 \end{aligned} \tag{17}$$ {GB2}

with

$$\begin{aligned} \mathbf{a}_1 &= \beta_1 \mathbf{n} + \mathbf{a}_{1t}, \qquad & \mathbf{a}_2 &= \beta_2 \mathbf{n} + \mathbf{a}_{2t} \\ \mathbf{b}_1 &= \alpha_1 \mathbf{n} + \mathbf{b}_{1t}, \qquad & \mathbf{b}_2 &= \alpha_2 \mathbf{n} + \mathbf{b}_{2t} \end{aligned} \tag{18}$$ {GB2b}

The wave vector for matched waves for this boundary obeys the condition

$$\mathbf{k}^i \cdot \left(\mathbf{k}^i \times \mathbf{b}_1 - k_0 \mathbf{a}_1\right) \times \left(\mathbf{k}^i \times \mathbf{b}_2 - k_0 \mathbf{a}_2\right) = 0 \tag{19}$$ {mw1}

where the free-space wave number is $k_0 = \omega\sqrt{\mu_0\epsilon_0}$. Likewise, a reflected wave can be matched to the boundary, whence we require a similar condition for the reflected wave vector $\mathbf{k}^r$:

$$\mathbf{k}^r \cdot (\mathbf{k}^r \times \mathbf{b}_1 - k_0\mathbf{a}_1) \times (\mathbf{k}^r \times \mathbf{b}_2 - k_0\mathbf{a}_2) = 0 \qquad (20) \quad \texttt{\{mw1b\}}$$

Matched waves at impedance boundary

For the special case of an impedance boundary with impedance dyadic $\overline{\overline{\mathsf{Z}}}_s$ in Equation (5), the condition for matched wave becomes

$$\eta_0 \left(\overline{\overline{\mathsf{Z}}}_s; \mathbf{k}_t\mathbf{k}_t + k_n^2 \, \mathrm{tr}\overline{\overline{\mathsf{Z}}}_s\right) = \pm k_0 k_n \left(\eta_0^2 + \det{}_t \overline{\overline{\mathsf{Z}}}_s\right) \qquad (21) \quad \texttt{\{mwIBC\}}$$

where tr is the trace and $\det_t$ is the two-dimensional determinant of the dyadic. Here the double-dot product is a short-hand notation for the trace of the dyadic product where one of the dyadics is transposed:

$$\overline{\overline{\mathsf{A}}} : \overline{\overline{\mathsf{B}}} = \mathrm{tr}\left(\overline{\overline{\mathsf{A}}} \cdot \overline{\overline{\mathsf{B}}}^T\right) = \mathrm{tr}\left(\overline{\overline{\mathsf{A}}}^T \cdot \overline{\overline{\mathsf{B}}}\right) \qquad (22) \quad \texttt{\{dd\}}$$

Matched waves at isotropic impedance boundary

The simplest IBC is the isotropic boundary with symmetric impedance dyadic

$$\overline{\overline{\mathsf{Z}}}_s = Z_s \overline{\overline{\mathsf{I}}}_t \qquad (23) \quad \texttt{\{iso\}}$$

Equation (21) allows four solutions, which in this case read

$$\frac{k_n}{k_0} = \pm\frac{Z_s}{\eta_0} \qquad \text{(parallel polarization)} \qquad (24) \quad \texttt{\{GB2c\}}$$

$$\frac{k_n}{k_0} = \pm\frac{\eta_0}{Z_s} \qquad \text{(perpendicular polarization)} \qquad (25) \quad \texttt{\{GB2d\}}$$

Here the upper signs correspond to a matched incident wave and the lower signs to a matched reflected wave. This is in agreement with the direction of the incident wave: for positive k_n (see Figure 2), a matched incident wave is absorbed into the boundary, requiring a positive real part of the surface impedance Z_s which corresponds to a dissipative boundary. And vice versa for the matched reflected wave which requires an active boundary and a negative surface resistance.

Because for real wave vectors, the incidence angle θ^i is related to the components of the wave vector $\mathbf{k}^i$ as $k_n = k_0 \cos\theta^i$, the parallel-polarized (TM) wave can be matched for $Z_s < \eta_0$ and the perpendicular polarization (TE) for $Z_s > \eta_0$. It is also geometrically obvious that for the parallel (perpendicular) polarization, the ratio of the tangential component of the electric (magnetic) field to the total field is k_n/k_0 which is in agreemnt with the boundary condition $\mathbf{E}_t = Z_s \, \mathbf{n} \times \mathbf{H}_t$.

Matched waves at PEMC boundary

Another isotropic boundary is the PEMC boundary with antisymmetric impedance dyadic. For PEMC boundary, the condition between the tangential fields is $\mathbf{H}_t = -M\mathbf{E}_t$, meaning that

$$\mathbf{E}_t = \overline{\overline{\mathsf{Z}}}_s \cdot (\mathbf{n} \times \mathbf{H}_t) \quad \text{with} \quad \overline{\overline{\mathsf{Z}}}_s = (1/M)\,\mathbf{n} \times \overline{\overline{\mathsf{I}}} = (1/M)\,\overline{\overline{\mathsf{J}}} \tag{26}$$ {PEMC5}

This leads to the following condition for matched waves:

$$k_n \left(\eta_0^2 + \frac{1}{M^2} \right) = 0 \tag{27}$$ {mwPEMC}

Equation (27) allows two types of solutions. The first one is a lateral wave: $k_n = 0$ and $\mathbf{k}^i = \mathbf{k}_t$. The polarization of such a wave is along the vector

$$\mathbf{E} \propto \mathbf{n} + \frac{\mathbf{n} \times \mathbf{k}}{k_0\, M\eta_0} \tag{28}$$ {mwE1}

The second solution for (27) is $M\eta_0 = \pm \mathrm{j}$. This is a particular case in the sense that it forces no limitation for the wave vector. It only fixes the magnitude of the PEMC parameter which is complex-valued. This is in agreement with the matched being either absorbed (dissipative boundary) or existient without incidence (active boundary), depending on the sign of the imaginary part of M.

The polarization of these matched waves is

$$\mathbf{E} \propto k_t^2 \mathbf{n} \mp \mathrm{j}k_0 \mathbf{n} \times \mathbf{k} + k_n \mathbf{k}_t \tag{29}$$ {mwE2}

where $k_t^2 = \mathbf{k} - t \cdot \mathbf{k}_t$. Due to the fact that the vector in (29) satisfies $\mathbf{E} \cdot \mathbf{E} = 0$, it is circularly polarized.

Matched waves at PEC and PMC boundaries

The PEC and PMC boundaries can be seen as special cases of both the isotropic symmetric boundary (23) and the isotropic antisymmetric PEMC boundary (26). In the first case, PEC and PMC correspond to $Z_s = 0$ and $Z_s^{-1} = 0$, respectively, and in the latter case they arise from $M^{-1} = 0$ and $M = 0$, respectively.

Hence also the matched waves appear from the previous results. Assuming homogeneous plane waves ($\mathbf{k}^i$ and $\mathbf{k}^r$ real-valued vectors), we can see that Equation (24) gives $k_n = 0$ for $Z_s = 0$ for the PEC boundary. This is a parallel-polarized lateral wave traveling along the boundary, in full agreement with Equation (28) for $M^{-1} = 0$, leading to the polarization $\mathbf{E} \propto \mathbf{n}$. Analogously, the PMC boundary supports a perpendicularly polarized matched wave, seen from (25) for $Z_s^{-1} = 0$ and from (28) for $M = 0$.

Note that matched waves can also be inhomogeneous plane waves, like for example in cases where $k_n > k_0$, thereby corresponding to leaky or surface waves.

4 Scattering by spheres with boundary condition

Another interesting area of electromagnetics problems in which boundary conditions play a significant role is scattering by objects with a given surface characterization. Let us consider here plane-wave scattering from a sphere as function of the boundary condition assigned on its surface.

A rigorous full-wave solution of the scattering problem by an isotropic homogeneous dielectric sphere has a long history starting from the works of L.V. Lorenz [24] and G. Mie [25] over a hundred years ago.[2] Excellent treatments of the analysis and computation of scattering by penetrable spheres can be found in textbooks [27, 28, 29], and the analysis has been applied to not only dielectric but also magnetic, plasmonic, chiral, and multilayer spherical scatterers.

The scattering problem of spheres with boundary conditions has received less attention. Apart from early treatments on PEC bodies, boundary-condition scatterers have been analyzed only recently. Let us review some of these results.

4.1 Scattering by PEC, DB, and D'B' spheres

When a plane wave hits an object, part of its forward-propagating energy will be lost. Two mechanisms are responsible for this loss: energy till be scattered to directions around the object, and also some part of the energy may be absorbed into the object which happens in the case of dissipative scatterers. The measure for the magnitude of these losses is the cross section. The scattering cross section of an object is equal to the area from which one needs to collect the power density of the incident wave that this power equals the total scattered power in all directions. In a similar manner, the absorption cross section is defined. The dimension of cross section is hence area and the unit is m^2. For a general scatterer, the cross sections are functions of (in addition to the size, shape, and constitution of the scatterer itself) the incident wave direction, its polarization and frequency.

Often the cross sections are normalized by the geometric cross section of the object, which leads to useful dimensionless quantities called *efficiencies*: the scattering efficiency $Q_{\rm eff}$, the absorption efficiency $Q_{\rm abs}$, and the extinction efficiency $Q_{\rm ext}$ which is the sum of the two other efficiencies:

$$Q_{\rm ext} = Q_{\rm sca} + Q_{\rm abs} \qquad (30)$$ {Q}

For spherical scatterers, the efficiencies can be computed as summation of the electric and magnetic multipole contributions, each of these so-called Lorenz–Mie scattering terms being rational expansions that contain spherical Bessel and spherical Hankel functions. Depending on the size of the sphere, a certain number of these terms need to be computed in order to have an accurate estimate of the efficiencies. The Wiscombe criterion [30]

$$N = x + 4x^{1/3} + 2 \qquad (31)$$ {Wis}

[2] For a review of the history of Lorenz–Mie scattering, see [26, Section 1.1].

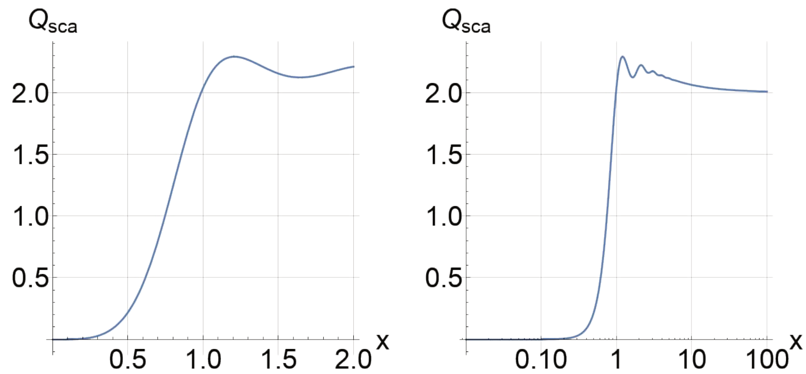

Figure 5: The scattering efficiency of a PEC sphere as function of its size parameter x, on linear and logarithmic size parameter dependence. {fig:PEC}

is commonly applied to estimate the necessary number of multipole terms in the expansions. Here $x = k_0 a$ is the size parameter for the sphere with radius a.[3]

A PEC scatterer is lossless. Hence its absorption efficiency vanishes, and the extinction equals scattering. Figure 5 displays the behavior of $Q_{\text{sca}} = Q_{\text{ext}}$ as function of its size parameter. In the region $x \ll 1$, the PEC sphere is a weak Rayleigh scatterer [31] whose efficiency is proportional to x^4. For high frequencies, the extinction cross section approaches the value of twice the geometrical cross section, which property is valid for all scatterers acconrding to the so-called extinction paradox [32].

The scattering efficiency only takes into account the intergrated scattered energy into different directions, and not the details of the scattering patterns in different scattering planes. Hence, due to duality between electric and magnetic fields, the scattering efficiency of a PMC sphere follows the same curve as the one for PEC in Figure 5. In fact, this is also valid for the PEMC sphere with any PEMC admittance parameter M.

However, the scattering characteristics of DB (Equation 9) and D'B' (Equation 10) boundary spheres are different than the PEC/PMC cases. These are shown in Figure 6. These type of scatterers show a larger dynamics than the rather smooth efficiency curve of the PEC sphere: the maximum values of Q_{sca} are higher: 2.80 for DB, 3.77 for D'B' while the largest value for PEC/PMC is 2.29. Another interesting observation is that in the Rayleigh scattering region (small x values), the D'B' sphere scatters clearly more strongly than the DB sphere.

[3] The Wiscombe criterion is fairly strict: for example to estimate the scattering efficiency of a PEC sphere with size parameter $x = 1$, the number of required terms using (31) is 7. This leads to a tremendous accuracy: a seven-term approximation of Q_{sca} leads to a number for which the relative error is less than 10^{-26}!

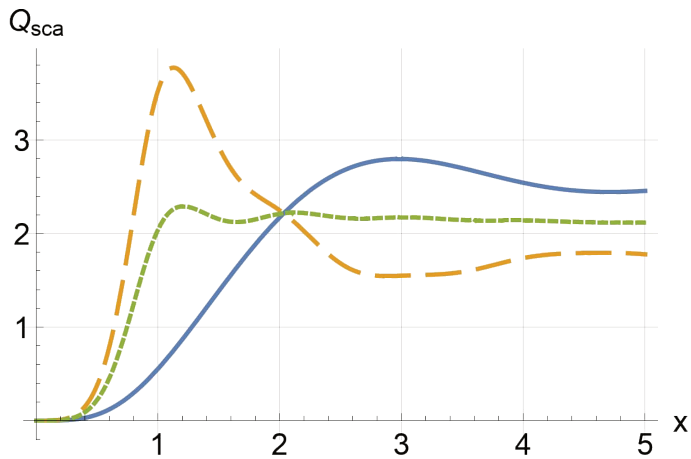

Figure 6: The scattering efficiency of DB (solid blue), D'B' (dashed orange), and PEC/PMC (dotted green) spheres as functions of the size parameter of the sphere x. {fig:D}

4.2 Scattering by impedance boundary spheres

Even if the curves for the scattering efficiency in Figures 5 and 6 were rather smooth, the situation is utterly different for spheres with impedance boundary condition. Let us consider the scattering efficiencies of spheres with isotropic symmetric impedance boundary dyadic $\overline{\overline{\mathsf{Z}}}_s = Z_s \overline{\overline{\mathsf{I}}}$, in other words, the tangential fields on the surface of the sphere are connected by

$$\mathbf{E}_t = Z_s\, \mathbf{n} \times \mathbf{H}_t \tag{32}$$ {isosphere}

The scattering, absorption, and extinction efficiencies can be computed using the modified Mie scattering coefficients [33], and now they are functions of not only the electrical size of the sphere x but also on the impedance scalar Z_s. The surface impedance can be split to the real part (surface resistance R_s) and the imaginary part (surface reactance X_s):

$$Z_s = R_s + \mathrm{j}\, X_s \tag{33}$$ {rx}

For lossless scatterers, the impedance is purely imaginary: $R_s = 0$. Such scatterers behave interestingly. Figure 7 shows the drastic effect of the surface reactance X_s on the scattering efficiency. While for the values $X_s = 0$ and $X_s^{-1} = 0$ (PEC and PMC), the smooth curve of Figure 5 is reproduced, very sharp resonances start to develop for small negative X_s values (capacitive reactances). These peaks grow larger and narrower as $|X_s|$ decreases, and at the same time, the size x decreases at which the resonance takes place.

Another illustration of the wild behavior of Q_{sca} of lossless impedance scatterers is shown in Figure 8. The contour plot of the efficiency as function of the

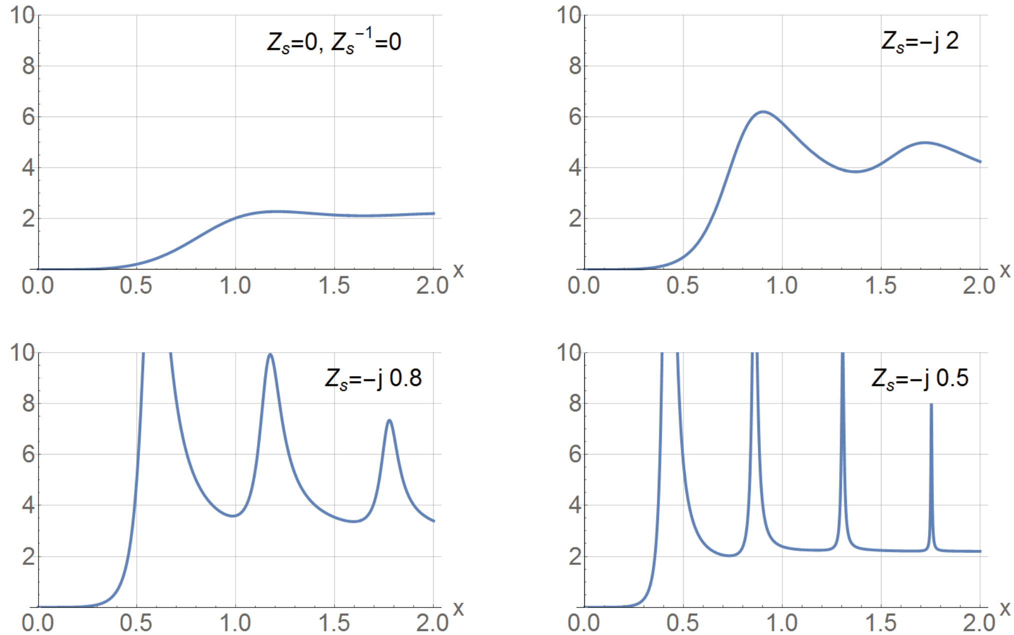

Figure 7: The scattering efficiency of lossless impedance-boundary spheres as functions of the size parameter of the sphere x for different surface reactances $\mathrm{Im}\{Z_s\}$. {fig:IBC1}

size parameter x and the surface reactance X_s (positive for inductive surfaces, negative for capacitive surfaces) shows the landscape of the resonances.

Indeed, as seen already in Figure 7, there appear ever-sharper peaks in the negative reactance range that accumulate towards small-x, small-X_s range. However, at the same time another range of resonances can be observed in the positive X_s domain. These inductive resonances move into larger and larger X_s values as the size of the scatterer decreases. Between these two "mountain ranges", the vertical line at $X_s = 0$ is seen to form a peaceful valley. This is the efficiency function of the PEC sphere.

It is counterintuitive that a small sphere would display a resonant electromagnetic behavior. However, this phenomenon is also known in plasmonic studies. Penetrable homogeneous spheres with negative permittivity (like silver nanoparticles) also can support localized surface plasmons which can be very sharp even if the particle itself is deeply subwavelength [34].

Qualitative differences exist, however, between the plasmonic resonances and those in the impedance spheres. The lozalized resonances in plasmonic nanoparticles are due to only electric multipoles (dipole, quadrupole, etc.). On the other hand, in Figure 8, the resonances clustering towards zero are of magnetic type, the most dominant being the magnetic dipole. Electric multipoles also form

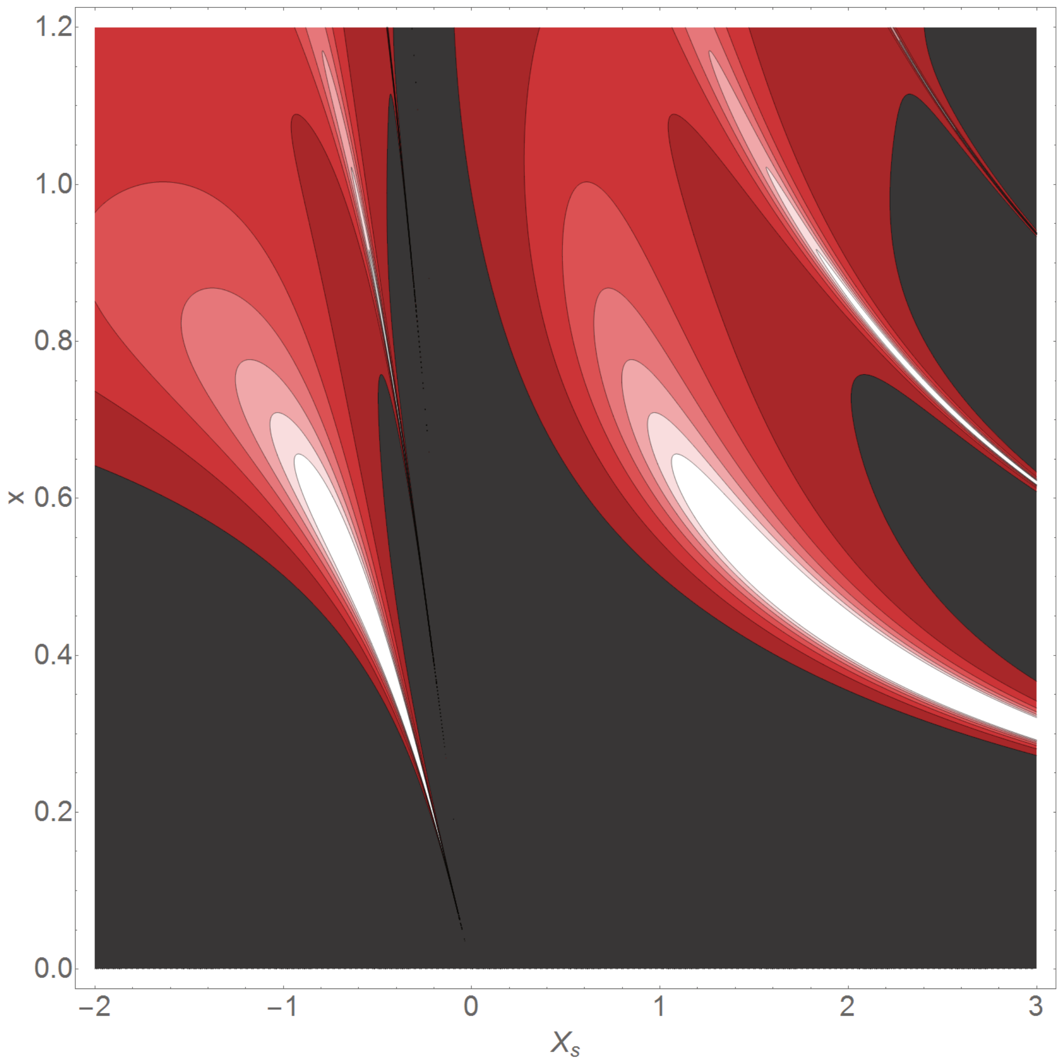

Figure 8: Contour plot of scattering efficiency of lossless impedance-boundary spheres as functions of the size parameter of the sphere x and the surface reactance $\text{Im}\{Z_s\}$. Dark regions correspond to small values of Q_{sca} while the white ones show the highest amplitudes. {fig:con1}

resonances in impedance spheres: these are due to the second tier that turns towards large positive X_s values in Figure 8.

5 Conclusion

The focus in the present article has been in the variety of boundary conditions in electromagnetics that extends far beyond the classical perfect electric conductor or perfect magnetic conductor cases. We have highlighted some very fundamental boundary conditions, like for example the perfect electromagnetic conductor (PEMC) condition and conditions that bind the normal components of the electric and magnetic fields, like the DB and D'B' boundaries. Hybrid boundary conditions that combine the tangential and normal components of fields form another large class of interesting conditions with examples like SHDB and E boundaries.

It is important to note that while many of the boundary conditions presented here have their inspiration from theoretical conderations and may seem even esoteric on the first sight, many of those have found experimental realizations. For example, the PEMC boundary has been demonstrated by graphene–ferrite metamaterial structures [35, 36], and in references [37, 38], the practical fabrication of a DB surface has been reported.

The concept of a matched wave is an essential underlying concept in connection with boundary conditions. It means a wave–propagating either into or away from the boundary—that by itself satisfies the boundary condition. Hence no reflection is generated, or no incidence is required. The properties of matched waves were discussed in connection with the most important boundary conditions.

Finally, a fascinating phenomenon of resonant structure in the scattering spectrum in subwavelength impedance-boundary scatterers was analyzed. The origin of the resonance modes leads to the poles of the magnetic and electric multipole coefficients and is especially visible for lossless scatterers. Like in the case of localized surface plasmons in negative-permittivity penetrable particles, these sharp resonance modes in impedance-boundary spheres can be supported by subwavelength objects. Obviously, the richness of such electromagnetic responses may be immensely amplified once the surface of these scatterers will be endowed with more complicated boundary conditions.

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