
This is an electronic reprint of the original article.
This reprint may differ from the original in pagination and typographic detail.

Volovik, G. E.

From elasticity tetrads to rectangular vielbein

Published in:
Annals of Physics

DOI:
[10.1016/j.aop.2022.168998](https://doi.org/10.1016/j.aop.2022.168998)

Published: 01/12/2022

Document Version
Publisher's PDF, also known as Version of record

Published under the following license:
CC BY

Please cite the original version:
Volovik, G. E. (2022). From elasticity tetrads to rectangular vielbein. *Annals of Physics*, 447, Article 168998. Advance online publication. <https://doi.org/10.1016/j.aop.2022.168998>

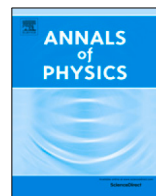
This material is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.



ELSEVIER

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

From elasticity tetrads to rectangular vielbein

G.E. Volovik*



Low Temperature Laboratory, School of Science and Technology, Aalto University, P.O. Box 15100, FI-00076 Aalto, Finland
 L.D. Landau Institute for Theoretical Physics, Kosygina 2, 119334 Moscow, Russia

ARTICLE INFO

Article history:

Received 3 June 2022

Received in revised form 27 June 2022

Accepted 27 June 2022

Available online 29 June 2022

Keywords:

Tetrad

Vielbein

Elasticity theory

Einstein–Cartan gravity

Dirac fermions

Superfluid ^3He

ABSTRACT

The paper is devoted to the memory of Igor E. Dzyaloshinsky. In our common paper Dzyaloshinskii and Volovick (1980), we discussed the elasticity theory described in terms of the gravitational field variables – the elasticity vielbein E_a^μ . They come from the phase fields, which describe the deformations of crystal planes. The important property of the elasticity vielbein E_a^μ is that in general they are not the square matrices. While the spacetime index μ takes the values $\mu = (0, 1, 2, 3)$, in crystals the index $a = (1, 2, 3)$, in vortex lattices $a = (1, 2)$, and in smectic liquid crystals there is only one phase field, $a = 1$. These phase fields can be considered as the spin gauge fields, which are similar to the gauge fields in Standard Model (SM) or in Grand Unification (GUT).

On the other hand, the rectangular vielbein e_a^μ may emerge in the vicinity of Dirac points in Dirac materials. In particular, in the planar phase of the spin–triplet superfluid ^3He the spacetime index $\mu = (0, 1, 2, 3)$, while the spin index a takes values $a = (0, 1, 2, 3, 4)$. Although these (4×5) vielbein describing the Dirac fermions are rectangular, the effective metric $g^{\mu\nu}$ of Dirac quasiparticles remains $(3+1)$ -dimensional. All this suggests the possible extension of the Einstein–Cartan gravity by introducing the rectangular vielbein, where the spin fields belong to the higher groups, which may include SM or even GUT groups.

© 2022 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

* Correspondence to: Low Temperature Laboratory, School of Science and Technology, Aalto University, P.O. Box 15100, FI-00076, Aalto, Finland.

E-mail address: grigori.volovik@aalto.fi.

<https://doi.org/10.1016/j.aop.2022.168998>

0003-4916/© 2022 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The paper Ref. [1] was published in the JETP issue dedicated to Dzyaloshinskii's 90th birthday. In that paper I discussed the application of the so-called elasticity vielbein, which were introduced in our common paper [2], to gravity.

The important property of the elasticity vielbein is that they have dimension of inverse length. If the gravitational tetrads have also this property, then the metric field $g_{\mu\nu}$, which is the bilinear combination of the tetrad fields, acquires the dimension of the inverse square of length $[g_{\mu\nu}] = 1/[l]^2$. This is distinct from the conventional dimensionless metric, $[g_{\mu\nu}] = 1$, in general relativity. The same dimension of the metric field, $[g_{\mu\nu}] = 1/[l]^2$, appears in the Diakonov theory [3–6], in which the tetrad itself is the bilinear combination of the fermionic fields, and thus the metric is the quadruplet of the fermions. The analog of the Diakonov scenario takes place in one of topological phases of superfluid ^3He – the B-phase [7].

Due to the change of the dimensionality of the metric, the physical quantities, which obey diffeomorphism invariance, become dimensionless. This takes place for such quantities as Newton constant, the scalar curvature, the cosmological constant, particle masses, fermionic and scalar bosonic fields, etc. This does not require the existence of the fundamental length, such as Planck length scale. The vacuum can be considered as superplastic (malleable) medium [8], in which the size of the elementary cell can be made arbitrary small.

Here I will discuss another interesting property of the elasticity vielbein. As distinct from the conventional gravitation tetrads, the elasticity vielbein is not necessary the square matrix. I will show several examples of the rectangular vielbein emerging in different systems, including Weyl and Dirac materials. When applied to gravity, this suggests that the gravity for fermions and bosons can be essentially different. While the fermions can be described by the $4 \times k$ vielbein matrix e_a^μ with $k \geq 4$, the bosonic fields see only the conventional 4×4 metric $g_{\mu\nu}$, which is the bilinear form of these rectangular vielbein fields [9].

2. Rectangular vielbein in elasticity theory

Elasticity vielbein is a convenient way to discuss non-linear elastic deformations of crystals in terms of the curved space [2] (the analog gravity in the elasticity theory of crystals see also in Refs. [10–15]). In the three-dimensional crystals, the vielbein E_μ^a with $a = 1, 2, 3$ form three elastic $U(1)$ gauge fields, which reflect the periodicity in crystals and three sets of crystal planes. As the gauge fields they have dimension of the inverse length, which is different from the dimensionless conventional tetrads in general relativity. The vielbein E_μ^a provide the proper description of different anomalous topological phenomena in topological insulators [16–19]. In particular, being the gauge fields, they together with the electromagnetic $U(1)$ gauge field A_μ enter the mixed topological Chern–Simons terms in the action.

The deformed crystal structure of the 3d crystal can be described as a system of three crystallographic surfaces of constant phase $X^a(x) = 2\pi n^a$, $n^a \in Z$ with $a = 1, 2, 3$. The intersection of the surfaces

$$X^1(\mathbf{r}, t) = 2\pi n^1, \quad X^2(\mathbf{r}, t) = 2\pi n^2, \quad X^3(\mathbf{r}, t) = 2\pi n^3, \quad (2.1)$$

are the nodes of a deformed crystal lattice. The elasticity vielbein are the gradients of the three phase functions:

$$E_\mu^a(x) = \partial_\mu X^a(x). \quad (2.2)$$

In an equilibrium crystal lattice the quantities E_i^a are lattice vectors of the reciprocal Bravais lattice. In a deformed crystal, but in the absence of dislocations the vielbein E_μ^a satisfy the integrability condition:

$$\nabla_\nu E_\mu^a - \nabla_\mu E_\nu^a = 0, \quad (2.3)$$

i.e. their field strength $F_{\mu\nu}^a = 0$. The nonzero field strength – the torsion field – appears in the presence of dislocations.

Note that the vielbein E_μ^a are non-square. While the space–time index μ is the 3+1-dimensional, $\mu = 0, 1, 2, 3$, the phase space index a takes values $a = 1, 2, 3$. This gives rise to the rectangular 4×3 vielbein E_μ^a .

The rectangular vielbein appears also in the dynamics of the other crystalline structures, for example in dynamics of the vortex lattice in superfluids and superconductors. The vortex lattice has two phase functions $X^1(\mathbf{r}, t) = 2\pi n^1$ and $X^2(\mathbf{r}, t) = 2\pi n^2$, corresponding to two sets of crystal planes, and is described by the 4×2 vielbein [20]. The dynamics of the smectic liquid crystals (see e.g. Ref. [21] and of the vortex sheets [22], which have single set of planes, is described by single phase function $X^1(\mathbf{r}, t) = 2\pi n^1$ and thus by the 4×1 vielbein. The periodically driven system (the Floquet crystal) serves as an exclusion: the vielbein becomes quadratic, forming the 4×4 tetrads [23].

3. Dirac fermions

3.1. Weyl fermions and tetrads

The quadratic 4×4 vielbein – tetrads – emerge in the vicinity of the Weyl points in Weyl semimetals and Weyl superfluids/superconductors [24,25]. The reason for that is that the Weyl point is the topological object, which can be represented as a Dirac magnetic monopole in the momentum space – a source or sink of Berry flux [24]. In the vicinity of the monopole, the inverse Green's function of fermions can be written in terms of tetrads e_a^μ :

$$G^{-1}(p_\mu) = \gamma^a e_a^\mu p_\mu, \quad (3.1)$$

where the γ -matrices are:

$$\gamma^0 = 1, \quad a = 0, \quad (3.2)$$

$$\gamma^a = \sigma^a, \quad a = 1, 2, 3. \quad (3.3)$$

Here p_0 is the (quasi)particle energy; p_i is the momentum counted from the Weyl point; and σ^a for $a = 1, 2, 3$ are the Pauli matrices, which correspond to the effective spin in the two-band structure.

In the Weyl materials, tetrads e_a^μ serve as the analog of the gravitational tetrads. The energy spectrum of the massless (gapless) fermions can be expressed in terms of the effective metric field [26]:

$$g^{\mu\nu} p_\mu p_\nu = 0, \quad (3.4)$$

where the metric $g^{\mu\nu}$ is the bilinear combination of the tetrads:

$$g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab}, \quad (3.5)$$

$$\eta^{ab} = \text{diag}(-1, 1, 1, 1). \quad (3.6)$$

Here the role of the phase space is played by the space of the effective spin. This phase space has the same dimension as the (orbital) coordinate spacetime. Both are governed by the Lorentz groups, $SO(1, 3)_S$ for the spin space and $SO(1, 3)_O$ for the spacetime.

Note also that while the vielbein E_μ^a , which emerges in the elasticity theory, has the covariant spacetime index μ , the tetrads e_a^μ emerging in the Dirac materials are the contravariant vectors. This demonstrates different origin of the tetrads in these two systems: geometric and dynamical. The elasticity tetrads E_μ^a are related to the geometry of crystal planes in coordinate space, while the Weyl tetrads are related to the topology in momentum space and thus to dynamics of Weyl fermions.

3.2. One-dimensional Dirac fermions with rectangular vielbein

The non-square vielbein e_a^μ can be obtained by decreasing the spacetime dimension without decreasing the spin space. For example, in the $1 + 1$ dimensional spacetime the inverse Green's function in Eq. (3.1) is expressed in terms of the 2×4 vielbein e_a^μ :

$$G^{-1}(p_0, p_x) = \gamma^a e_a^\mu p_\mu = \gamma^a (e_a^0 p_0 + e_a^1 p_x), \quad (3.7)$$

The energy spectrum of these massless fermions can be expressed in terms of the metric field:

$$g^{\mu\nu} p_\mu p_\nu = 0, \quad \mu, \nu = (0, 1), \tag{3.8}$$

$$g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab}, \quad a, b = (0, 1, 2, 3), \tag{3.9}$$

$$\eta^{ab} = \text{diag}(-1, 1, 1, 1). \tag{3.10}$$

Eq. (3.7) describes the 1+1 dimensional Dirac point (or in the other language the 0-dimensional Fermi surface). In the simple example, when $e_1^0 = e_2^0 = e_3^0 = e_0^1 = 0$, one obtains the energy spectrum:

$$p_0 = \pm v_F p_x, \tag{3.11}$$

$$v_F = \frac{|\mathbf{e}^1|}{e_0^0}, \quad \mathbf{e}^1 = (e_1^1, e_2^1, e_3^1). \tag{3.12}$$

Here v_F is the Fermi velocity, and the 3d spin vector \mathbf{e}^1 determines the axis of spin quantization.

The relativistic Dirac spectrum in Eq. (3.11) does not depend on the number of components of the vector \mathbf{e}^1 , or in general it does not depend on the dimension of the phase (spin) space. Moreover, when the spectrum of Dirac fermions is expressed in terms of the effective metric in Eq. (3.8), the dimension of spin space is lost. The (1 + 1)-dimensional metric $g^{\mu\nu}$, which describes the Dirac fermions, has no information on the (3+1) dimension of the spin space.

Another property of the rectangular vielbein concerns the metric $g_{\mu\nu}$, which enters the effective interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{3.13}$$

This metric is obtained as inverse to $g^{\mu\nu}$, but it cannot be obtained from the inverse vielbein, since the non-square vielbein matrix e_a^μ is not invertible. Whether the Moore–Penrose generalization of the inverse for rectangular matrices [27,28] (pseudoinverse) can be useful for the physics of vielbein is an interesting question.

All this suggests that the same can be applied to Dirac and Weyl fermions in (3+1)-dimensional spacetime. Instead of the 4×4 tetrads in Section 3.1, the Dirac and Weyl fermions can be described by the rectangular $4 \times k$ vielbein with $k > 4$. Dirac fermions with 4×5 vielbein emerge in the planar phase of the spin-triplet p -wave superfluids/superconductors [9], which we shall discuss in Section 4.

4. Rectangular vielbein in planar phase of the spin-triplet superfluid

In the general spin triplet p -wave pairing state, the 2×2 matrix of the gap function is:

$$\hat{\Delta}(\mathbf{p}) = A_\alpha^i \sigma^\alpha p_i, \tag{4.1}$$

where σ^α are the Pauli matrices for spin and A_α^i is the 3×3 complex matrix of the order parameter, see the book [29].

The Bogoliubov–Nambu Hamiltonian for quasiparticles is the 4×4 matrix, which includes the conventional spin with σ -matrices and the Bogoliubov–Nambu particle–hole isospin with τ -matrices:

$$\begin{pmatrix} \epsilon(p) & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\epsilon(p) \end{pmatrix}, \tag{4.2}$$

where $\epsilon(p) = c_\parallel(p - p_F)$, $c_\parallel = v_F$, and v_F and p_F are correspondingly the Fermi velocity and Fermi momentum of the normal Fermi liquid.

In the planar phase the general form of the order parameter is:

$$A_\alpha^i = c_\perp e^{i\Phi} \left(R_\alpha^i - \hat{l}^i \hat{s}_\alpha \right). \tag{4.3}$$

Here Φ is the phase of the order parameter, which we shall ignore later; \hat{l} is the unit vector of the orbital anisotropy of the planar phase; R_α^i is the matrix of rotation from the orbital space to spin

space; $\hat{s}_\alpha = R_{\alpha k} \hat{l}^k$ is the unit vector of the uniaxial anisotropy in the spin space; and $p_F c_\perp$ is the gap amplitude.

The quasiparticle spectrum has two Dirac points at $\mathbf{p} = \mathbf{p}_{\text{Dirac}} = \pm p_F \hat{\mathbf{l}}$. To write the spectrum in the vicinity of the Dirac points, let us introduce the Dirac γ -matrices in the following way:

$$\gamma^0 = i\tau_2, \quad \gamma^a = \tau_3 \sigma^a \quad (a = 1, 2, 3), \quad \gamma^4 = \tau_1, \quad (4.4)$$

where σ^a and τ_a with $a = 1, 2, 3$ are two sets of Pauli matrices. Five γ -matrices in Eq. (4.4) obey anticommutation relations:

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \quad \eta^{ab} = \text{diag}(-1, 1, 1, 1, 1). \quad (4.5)$$

The properly determined inverse Green's function near one of the two Dirac points has the form of Eq. (3.1):

$$G^{-1}(p_\mu) = \gamma^a e_a^\mu p_\mu, \quad (4.6)$$

where the momentum p_μ is counted from the chosen Dirac point. The Green's function contains 4×5 vielbein. The nonzero vielbein components in the planar state in Eq. (4.3) at $\Phi = 0$ are

$$e_0^0 = 1, \quad (4.7)$$

$$e_a^i = c_\perp \left(R_\alpha^i - \hat{l}^i \hat{s}_\alpha \right) \quad \text{for } a = 1, 2, 3, \quad (4.8)$$

$$e_4^i = c_\parallel \hat{l}^i. \quad (4.9)$$

In spite of the (4×5) -dimensional vielbein describing these fermions, their energy spectrum is given by the conventional 3+1 metric in Eq. (3.4):

$$g^{\mu\nu} p_\mu p_\nu = 0. \quad (4.10)$$

This metric is expressed in terms of (4×5) vielbein:

$$g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab}, \quad \eta^{ab} = \text{diag}(-1, 1, 1, 1, 1). \quad (4.11)$$

With vielbein in Eqs. (4.7)–(4.9) the metric elements are:

$$g^{00} = -1, \quad g^{ik} = c_\parallel^2 \hat{l}^i \hat{l}^k + c_\perp^2 (\delta^{ik} - \hat{l}^i \hat{l}^k), \quad (4.12)$$

$$g_{00} = -1, \quad g_{ik} = \frac{1}{c_\parallel^2} \hat{l}_i \hat{l}_k + \frac{1}{c_\perp^2} (\delta_{ik} - \hat{l}_i \hat{l}_k). \quad (4.13)$$

It is interesting, that for $c_\parallel^2 = c_\perp^2 \equiv c^2$ the metric corresponds to the Minkowski vacuum, $g_{00} = -1$ and $g_{ik} = (1/c^2)\delta_{ik}$, even when the vielbein in Eqs. (4.7)–(4.9) depends on space and time. In particular, the vielbein may have the topologically nontrivial structure. For example, it may have the global monopole configuration of the $\hat{\mathbf{l}}$ -vector, $\hat{\mathbf{l}}(\mathbf{r}) = \hat{\mathbf{r}}$ [9], but the space-time remains Minkowski (except for the region of the core of the monopole, where $c(r=0) = 0$). This is distinct from the conventional global monopoles in general relativity, which give rise to the conical spacetime [30–38].

5. Massive Dirac field with rectangular vielbein

The massless Dirac fermions with (4×5) -dimensional vielbein can be extended to the massive case. Dirac fermions with (4×5) vielbein may arise in the Dirac materials when there are two successive phase transitions. The first transition produces the spin triplet p -wave phase corresponding to the planar phase in Section 4. In the second transition the Z_2 time reversal symmetry is broken, and the imaginary s -wave component appears in the gap function. Such $p + is$ state was also discussed in superconductors, see e.g. Refs. [39–41]. In the simplest case the planar phase order parameter in Eq. (4.3) is extended to

$$\hat{\Delta} = e^{i\Phi} \left(c_\perp \left(R_\alpha^i - \hat{s}_\alpha \hat{l}^i \right) p_i \sigma^\alpha + i\chi \right). \quad (5.1)$$

Here χ is the nonzero imaginary part of the s-wave component – the real scalar Higgs field describing the broken time reversal symmetry. On the Higgs fields in different superfluid and superconducting systems see review [42].

This *planar + is* phase has massive Dirac fermions. Near the points $\mathbf{p}_{\text{Dirac}} = \pm p_F \hat{\mathbf{f}}$ the proper inverse Green's function is

$$G^{-1}(p_\mu) = i\gamma^a e_a^\mu p_\mu + M. \tag{5.2}$$

Here the Dirac mass is $M = \chi$, while the (4×5) -dimensional vielbein e_a^μ remains the same as in the massless case, with $\mu = (0, 1, 2, 3)$ and $a = (0, 1, 2, 3, 4)$. The spectrum of massive Dirac particles is also determined by the same relativistic 3+1 metric:

$$g^{\mu\nu} p_\mu p_\nu + M^2 = 0. \tag{5.3}$$

The *planar + is* phase has the topologically stable global monopole in the same manner as in the pure planar phase. It is also the hedgehog in the $\hat{\mathbf{l}}$ -vector with $\hat{\mathbf{l}}(\mathbf{r}) = \pm \mathbf{r}$, and this monopole also does not disturb the metric if $c_{\parallel}^2 = c_{\perp}^2 \equiv c^2$. Such objects are impossible for the conventional Dirac, but they become possible, if the Dirac matrices or vielbein have nontrivial topology, i.e. the nontrivial dependence on spacetime. This suggests that the nontrivial topology of the γ -matrices can be the proper extension of general relativity.

Note the difference with the conventional Dirac equation. In the latter case the matrix γ^4 is actually the γ^5 matrix, which is responsible for the axial mass in the mass term $M_1 + M_2 \gamma^5$. In our case, γ^4 enters the kinetic term $\gamma^4 e_4^\mu p_\mu$, and the spin group is $SO(1, 4)_S$. The group $SO(1, 4)_S$ can be also obtained from the Lorentz group by extension of the discrete symmetry to the continuous symmetry [43] (how the discrete group Z_2 transforms to the continuous group $U(1)$ in the planar phase, is demonstrated in Ref. [44]).

6. Possible extension to GUT

The example of the planar phase suggests that 4×4 Dirac γ -matrices can be extended to the matrices with higher spin dimension without extension of the space dimension. In general, one may have the Lorentz $SO(1, N)_S$ group for spin space and the Lorentz $SO(1, d)_O$ group for (orbital) spacetime with $N > d$, and with Γ matrices obeying the Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}. \tag{6.1}$$

This results in the rectangular $(d + 1) \times (N + 1)$ vielbein e_a^μ , which enter the Green's function:

$$G^{-1}(p_\mu) = i\Gamma^a e_a^\mu p_\mu + M, \tag{6.2}$$

while the metric is the square $(d + 1) \times (d + 1)$ matrix:

$$g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab}. \tag{6.3}$$

This metric enters the conventional energy spectrum of massive particle in the $d + 1$ spacetime:

$$g^{\mu\nu} p_\mu p_\nu + M^2 = 0. \tag{6.4}$$

It is important that for $N > d$ the inverse vielbein are absent. The spin and spacetime indices are raised and lowered with η^{ab} and with metric $g^{\mu\nu}$ correspondingly. But now the spin matrix η_{ab} cannot have expression similar to Eq. (6.3) for metric:

$$\eta_{ab} \neq g_{\mu\nu} e_a^\mu e_b^\nu. \tag{6.5}$$

This can be checked multiplying both sides by η^{ab} :

$$\eta^{ab} \eta_{ab} = N + 1 \neq \eta^{ab} g_{\mu\nu} e_a^\mu e_b^\nu = g_{\mu\nu} g^{\mu\nu} = d + 1. \tag{6.6}$$

Another property of the rectangular vielbein with $N > d$ concerns the metric determinant. For $N = d$, the metric determinant is always negative, since $g = -e^2 < 0$, where e is the tetrad determinant. For the non-square vielbein matrix the determinant is absent, and thus there is no

constraint on the sign of the metric determinant. Let us consider the simplest case of the 4×5 vielbein, when the 4×4 tetrad part of the vielbein is diagonal, while the extra vielbein part e_4^μ contains only nonzero element e_4^0 . Then the metric elements are

$$g^{ik} = \delta^{ik}, \quad g^{00} = -(e_0^0)^2 + (e_4^0)^2, \quad (6.7)$$

which allows for the signature change.

As a result, the phase transition with the change of the sign of the determinant becomes possible. In this type of the tetrad gravity the metric signature becomes dynamical. The discussion of the dynamical metric signature can be found in Refs. [45,46].

The extension of the spin space from $1 + 3$ to $N + 3$ with $N > 3$ is essentially different from the extension of the spacetime. As distinct from string theories, the coordinate spacetime remains $3 + 1$ dimensional. The spin space extension has something common with the unification of GUT with Lorentz group [43]. Example is the group $SO(1, 11)$, which corresponds to $N = 11$. This group can be decomposed into the subgroups:

$$SO(1, 11) \sim SO(10) \times SO_F(6) \times SO(1, 3)_S. \quad (6.8)$$

Here $SO(10)$ is the GUT group; $SO_F(6) = SU_F(4)$ is the family group for 4 generations of fermions; and $SO(1, 3)_S$ is the Lorentz spin group. The number of generators of $SO(1, 11)$ group is $66 = 45 + 15 + 6$. The Clifford algebra of the corresponding Γ -matrices is $Cl_{11,1}(\mathbf{R})_c$. See also the IKKT (Ishibashi–Kawai–Kitazawa–Tsuchiya [47]) matrix model in string theory (or the type IIB matrix model) [48], where there is the $SO(1, 9)$ group with 10-dimensional Γ -matrices, and 16 fermionic variables.

In principle, the reduction from the $(d + 1) \times (N + 1)$ vielbein to $(d + 1) \times (d + 1)$ vielbein may correspond to the symmetry breaking (topological) phase transition $SO(1, N) \rightarrow SO(1, d)$.

7. Conclusion

The elasticity vielbein discussed in Refs. [2,16] come from the phase fields, which describe the deformations of crystal. These phase fields can be considered as the spin gauge fields, which are similar to the gauge fields in Standard Model or in Grand Unification. That is why the spin fields can be incorporated into the higher group, which may include both the spin group and the groups of SM or GUT. The same concerns the extension of the Lorentz spin group for Dirac fermions to the higher group, which may also include the SM or GUT groups.

Here we considered several examples of such extension, which demonstrated that the gravitational tetrads can be generalized to the non-square (rectangular) vielben, while the metric remains (3+1)-dimensional. This happens for the spin-triplet p -wave planar phase, where the Dirac quasiparticles are described by the (4×5) -vielbein, but by the (4×4) effective metric. The possible extension of the GUT theory is considered, which incorporates the $SO(1, 3)_S$ spin group.

How the rectangular vielben modifies the Einstein–Cartan equations of general relativity, can be the interesting problem. This especially concerns the spin connection and torsion fields.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

I thank Jaakko Nissinen for discussions. This work has been supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (Grant Agreement No. 694248).

References

- [1] G.E. Volovik, ZhETF 159 (2021) 815–821; JETP 132 (2021) 727–733, arXiv:2006.16821.
- [2] I.E. Dzyaloshinskii, G.E. Volovick, Ann. Physics 125 (1980) 67–97.
- [3] D. Diakonov, Towards lattice-regularized Quantum Gravity, arXiv:1109.0091.
- [4] A.A. Vladimirov, D. Diakonov, Phys. Rev. D 86 (2012) 104019.
- [5] A.A. Vladimirov, D. Diakonov, Phys. Part. Nuclei 45 (2014) 800.
- [6] Y.N. Obukhov, F.W. Hehl, Phys. Lett. B 713 (2012) 321–325.
- [7] G.E. Volovik, Physica B 162 (1990) 222–230.
- [8] F.R. Klinkhamer, G.E. Volovik, Pis'ma ZhETF 109 (2019) 369–370; JETP Lett. 109 (2019) 362–365, arXiv:1812.07046.
- [9] G.E. Volovik, Pis'ma ZhETF 112 (2020) 539–540; JETP Lett. 112 (2020) 505507, arXiv:2009.09779.
- [10] B.A. Bilby, E. Smith, Proc. Roy. Soc. Sect. A 231 (1955) 263–273.
- [11] B.A. Bilby, E. Smith, Proc. Roy. Soc. Sect. A 236 (1956) 481–505.
- [12] E. Kröner, Arch. Ration. Mech. Anal. 4 (1960) 18–334.
- [13] A.F. Andreev, M. Yu. Kagan, Zh. Eksp. Teor. Fiz. 86 (1984) 546; Sov. Phys.–JETP 59 (1984).
- [14] H. Kleinert, J. Zaanen, Phys. Lett. A 324 (2004) 361–365.
- [15] F.W. Hehl, Y.N. Obukhov, Ann. Fond. Louis Broglie 32 (2007) 157–194.
- [16] J. Nissinen, G.E. Volovik, Phys. Rev. Res. 1 (2019) 023007, arXiv:1812.03175.
- [17] Xue-Yang Song, Yin-Chen He, Ashvin Vishwanath, Chong Wang, Electric polarization as a nonquantized topological response and boundary Luttinger theorem, arXiv:1909.08637.
- [18] J. Nissinen, T.T. Heikkilä, G.E. Volovik, Phys. Rev. B 103 (2021) 245115, arXiv:2008.02158.
- [19] J. Nissinen, Field theory of higher-order topological crystalline response, generalized global symmetries and elasticity tetrads, arXiv:2009.14184.
- [20] G.E. Volovik, Vik. S. Dotsenko, Pisma ZhETF 29 (1978) 630633; JETP Lett. 29 (1978) 576–579.
- [21] M. Kleman, O. Parodi, J. Phys. 36 (1975) 671–681.
- [22] G.E. Volovik, Phys.-Usp. 58 (2015) 897–905, arXiv:1504.00336.
- [23] J. Nissinen, G.E. Volovik, ZhETF 154 (2018) 1051–1056, arXiv:1803.09234.
- [24] G.E. Volovik, Pisma ZhETF 46 (1987) 81–84; JETP Lett. 46 (1987) 98–102.
- [25] Horava P., Phys. Rev. Lett. 95 (2005) 016405.
- [26] J. Nissinen, G.E. Volovik, JETP Lett. 105 (2017) 447–452, arXiv:1702.04624.
- [27] R. Penrose, Proc. Cambridge Philos. Soc. 51 (1955) 406–413.
- [28] T.N.E. Greville, SIAM Rev. 8 (1966) 518–521.
- [29] D. Vollhardt, P. Wölfle, The Superfluid Phases of Helium 3, Taylor & Francis, London, 1990.
- [30] D.D. Sokoloff, A.A. Starobinskii, Dokl. Akad. Nauk SSSR 234 (1977) 1043–1046; Sov. Phys. Dokl. 22 (1977) 312.
- [31] M. Barriola, A. Vilenkin, Phys. Rev. Lett. 63 (1989) 341–343.
- [32] D. Harari, C. Loustó, Phys. Rev. D 42 (1990) 2626.
- [33] E.R. Bezerra de Mello, Braz. J. Phys. 31 (2001) 211.
- [34] K.A. Bronnikov, B.E. Meierovich, E.R. Podolyak, JETP 95 (2002) 392–403.
- [35] N.E. Mavromatos, S. Sarkar, Phys. Rev. D 95 (2017) 104025.
- [36] J.R. Nascimento, Gonzalo J. Olmo, P.J. Porfirio, A.Yu Petrov, A.R. Soares, Phys. Rev. D 99 (2019) 064053.
- [37] E.A.F. Braganca, R.L.L. Vitoria, H. Belich, E.R. Bezerra de Mello, Eur. Phys. J. C 80 (2020) 206.
- [38] N. Kaloper, Internat. J. Modern Phys. D 31 (2022) 2250035.
- [39] M. Smidman, M.B. Salamon, H.Q. Yuan, D.F. Agterberg, Rep. Progr. Phys. 80 (2017) 036501.
- [40] T. Shang, Phys. Rev. Lett. 124 (2020) 207001.
- [41] B. Roy, Phys. Rev. B 101 (2020) 220506 (R), arXiv:2003.12566.
- [42] G.E. Volovik, M.A. Zubkov, J. Low Temp. Phys. 175 (2014) 486–497, arXiv:1305.7219.
- [43] A. Maiezza, F. Nesti, Parity from gauge symmetry, arXiv:2111.11076.
- [44] Yu. Makhlin, M. Silaev, G.E. Volovik, Phys. Rev. B 89 (2014) 174502, arXiv:1312.2677.
- [45] S. Bondarenko, Dynamical signature: complex manifolds, gauge fields and non-flat tangent space, arXiv:2111.06095.
- [46] S. Bondarenko, M.A. Zubkov, Pis'ma ZhETF 116 (2022) #1; JETP Lett. 116 (2022) #1.
- [47] N. Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya, Nuclear Phys. B 498 (1997) 467–491.
- [48] J. Nishimura, Signature change of the emergent space-time in the IKKT matrix model, arXiv:2205.04726.