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Effective temperature pulses in open quantum systems

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Controlling the temperature of nanoscale quantum systems is becoming increasingly important in the efforts to develop thermal devices such as quantum heat valves, heat engines, and refrigerators and to explore fundamental concepts in quantum thermodynamics. In practice, however, it is challenging to generate arbitrary time-dependent temperatures, similarly to what has been achieved for electronic voltage pulses. To overcome this problem, we here propose a fully quantum mechanical scheme to control the time-dependent environment temperature of an open quantum system. To this end, we consider a collection of quantum harmonic oscillators that mediate the interactions between the quantum system and a thermal reservoir, and we show how an effective time-dependent temperature can be realized by modulating the oscillator frequencies in time. By doing so, we can apply effective temperature pulses to the quantum system, and it can be cooled below the temperature of the environment. Surprisingly, the scheme can be realized using only a few oscillators, and our proposal thereby paves the way for controlling the temperature of open quantum systems.

I. INTRODUCTION

Temperature is a central concept in thermodynamics, which characterizes the tendency of a physical system to emit or absorb heat [1]. Typically, temperatures vary only slowly in time; however, fast temperature modulations are playing an increasingly important role in quantum thermodynamics, for instance, in relation to finite-time thermal machines [2–7], fluctuation theorems [8–12], and quantum calorimetry [13–16]. Recently, the field of thermotronics has emerged with the goal of realizing thermal capacitors [17–20], transistors [21–23], and other nonlinear components such as thermal memristors [24,25] that are driven by time-dependent temperatures instead of voltages. However, fast control of the temperature at the nanoscale is challenging, and most proposals for temperature control [12,26–31] rely on semiclassical techniques or time-dependent couplings to environments at different temperatures [32,33].

Several approaches to temperature control are based either on random forces acting on a system [26,29,30,34] or on the use of feedback cooling (or heating) by a parametric drive [12,27,28,31]. As an example, the stiffness of an optical potential may be changed in response to the movement of a trapped particle. More generally, structured environments have been employed to generate non-Markovian dynamics, for instance, by using a finite-size system that mediates the interactions between an open quantum system and its environment [35]. Alternatively, a finite-size system can mimic

a macroscopic environment for a limited period of time [36,37]. Potentially, these methods may also be used to control the effective temperature of an environment and thereby
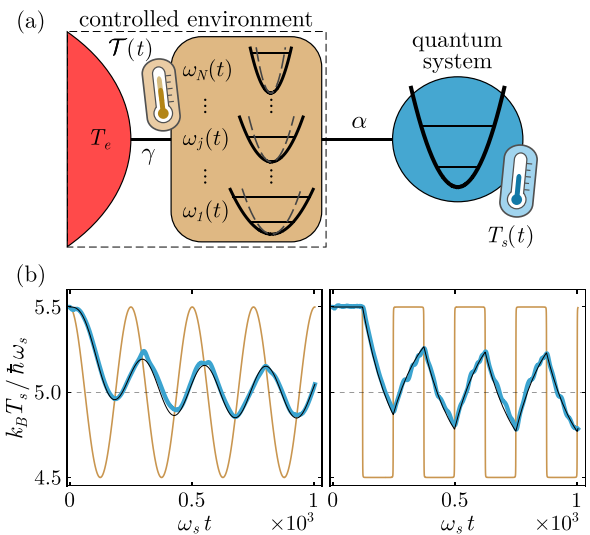

FIG. 1. Effective temperature pulses. (a) A quantum system (blue) is coupled with strength α to an environment consisting of N quantum harmonic oscillators (brown) that are coupled with the rate γ to a thermal reservoir (red) at the fixed temperature T_e. We modulate the oscillator frequencies, $\omega_j(t)$, to generate a time-dependent effective temperature $\mathcal{T}(t)$ that the quantum system experiences. (b) As an application, we take a single harmonic oscillator with the frequency ω_s as the quantum system, and we show its temperature $T_s(t)$ (blue) in response to the temperature pulses $\mathcal{T}(t)$ (brown) with $N = 16$. As a comparison, we show the time-dependent temperature obtained from a Lindblad equation for the quantum system only with $\mathcal{T}(t)$ inserted (black); see Eq. (8). The parameters are $\alpha = 0.05\nu\omega_s$, $\gamma = 0.01\omega_s$, and $k_B T_e = 5\hbar\omega_s$.

apply temperature pulses to an open quantum system as we will see.

In this paper, we propose and analyze a fully quantum mechanical scheme to control the effective temperature of an open quantum system. Figure 1(a) illustrates our setup, which consists of a collection of quantum harmonic oscillators that mediate the interactions between a small quantum system and a thermal reservoir, which is kept at a fixed temperature. Inspired by Luttinger, we modulate the frequencies of the oscillators in time to generate a time-dependent effective temperature that the quantum system is exposed to [38,39]. In particular, we determine how the oscillator frequencies must be modulated in time to realize a desired effective temperature profile $\mathcal{T}(t)$ as illustrated in Fig. 1(b), where we show how the temperature of the quantum system responds to two specific temperature drives using $N = 16$ quantum harmonic oscillators. This approach also allows us to cool the quantum system and maintain its temperature below that of the environment. Our work thereby provides a practical strategy for controlling the effective temperatures of open quantum systems, and it opens up avenues for further theoretical and experimental investigations of temperature-driven quantum systems.

II. EFFECTIVE TEMPERATURES

The Hamiltonian of the quantum system and the harmonic oscillators reads

$$\hat{H}(t) = \hat{H}_s + \hat{H}_a(t) + \hat{H}_i, \tag{1}$$

where the quantum system itself is a harmonic oscillator $\hat{H}_s = \hbar\omega_s(\hat{a}_s^\dagger\hat{a}_s + 1/2)$ that interacts with the other oscillators, $\hat{H}_a(t) = \sum_{j=1}^{N} \hbar\omega_j(t)(\hat{a}_j^\dagger\hat{a}_j + 1/2)$, through the coupling $\hat{H}_i = \hbar\alpha\sum_{j=1}^{N}(\hat{a}_s^\dagger\hat{a}_j + \hat{a}_j^\dagger\hat{a}_s)$, and $\hat{a}_{s,j}^\dagger$ and $\hat{a}_{s,j}$ are the usual ladder operators. Our scheme applies to other types of quantum systems, such as two-level systems; however, it is useful to consider an oscillator, since it can be described by an effective temperature that we denote by $T_s(t)$. The key idea is now to modulate the frequencies $\omega_j(t)$ in time to generate an effective temperature profile $\mathcal{T}(t)$ that the quantum system experiences.

The coupling between the quantum system and the oscillators is weak, $\alpha \ll \omega_s, \omega_j$. In addition, the oscillators are weakly coupled to a heat bath at the temperature T_e, and the driving is slower than the bath correlation time. The reduced density matrix $\hat{\rho}(t)$ of system and oscillators then evolves according to the Lindblad equation [40–42]

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}[\hat{H}(t), \hat{\rho}(t)] + \sum_{j=1}^{N} \gamma_j(\omega_j)\mathcal{D}_j\hat{\rho}(t), \tag{2}$$

where we have introduced the dissipators $\mathcal{D}_j\hat{\rho} = [1 + n_{T_e}(\omega_j)](\hat{a}_j\hat{\rho}\hat{a}_j^\dagger - \frac{1}{2}\{\hat{a}_j^\dagger\hat{a}_j, \hat{\rho}\}) + n_{T_e}(\omega_j)(\hat{a}_j^\dagger\hat{\rho}\hat{a}_j - \frac{1}{2}\{\hat{a}_j\hat{a}_j^\dagger, \hat{\rho}\})$, the rates γ_j depend on the oscillator frequencies via the spectral function of the heat bath, and $n_T(\omega) = [\exp(\hbar\omega/k_B T) - 1]^{-1}$ is the Bose-Einstein distribution. The derivation of Eq. (2) is provided in Appendix A together with the relevant assumptions. We note that the oscillator frequencies in the Hamiltonian may be slightly renormalized due to a small Lamb shift.

III. DRIVING PROTOCOL

To find the driving protocol for the oscillator frequencies $\omega_j(t)$ to realize a given temperature profile $\mathcal{T}(t)$, we first set the coupling to zero ($\alpha = 0$) and impose that the N oscillators remain in the thermal state

$$\hat{\rho}_a(t) = e^{-\hat{H}_a(t)/k_B \mathcal{T}(t)}/Z(t) \tag{3}$$

with $Z(t)$ defined so that $\text{tr}\{\hat{\rho}_a(t)\} = 1$. With $\mathcal{T}(t) = T_e$, this is the state of the oscillators in equilibrium with the reservoir. Generally, however, it is a nonequilibrium state, but it takes the form of an equilibrium density matrix with the effective temperature $\mathcal{T}(t)$. Inserting Eq. (3) into Eq. (2), we obtain the differential equation

$$\frac{d}{dt}n_{\mathcal{T}}(\omega_j) = \gamma(\omega_j)[n_{T_e}(\omega_j) - n_{\mathcal{T}}(\omega_j)] \tag{4}$$

for each oscillator frequency; see Appendix B. The left-hand side is the required change in the population of each oscillator to retain the state in Eq. (3), and the right-hand side is the net absorption rate of energy quanta from the environment. Without the reservoir ($\gamma_j = 0$), we easily find $\omega_j(t)/\omega(0) = \mathcal{T}(t)/\mathcal{T}(0)$, showing that the change in the frequencies is directly proportional to the desired temperature profile. Furthermore, for $\mathcal{T}(t) = T_e$, we find $\dot{\omega}_j = 0$, and no driving is needed, as one would expect.

To illustrate our proposal, we now consider the particular case where the rates γ_j are constant, corresponding to a heat bath with a flat spectral density [43–45]. Also, for the sake of simplicity, we take the same rate for all oscillators, $\gamma_j = \gamma$, although none of these assumptions is essential for our proposal. In the general case of a time-dependent effective temperature, $\mathcal{T}(t) \neq T_e$, we find the time dependence of the oscillator frequencies by solving Eq. (4) for $\omega_j(t)$. For large temperatures, $k_B T_e, k_B \mathcal{T}(t) \gg \hbar\omega_j(t)$, and small variations, $\mathcal{T}(t) \equiv T_0 + \Delta(t)$, we find the expression

$$\frac{\omega_j(t)}{\omega_j(0)} \simeq e^{(1-T_e/T_0)\gamma t} + \int_0^t d\tau(\gamma T_e\Delta(\tau) + T_0\dot{\Delta}(\tau))/T_0^2. \tag{5}$$

Here, as a special case, the integral vanishes for a constant effective temperature, $\mathcal{T}(t) = T_0$, and we are left only with the first term, which is exponentially decreasing (increasing) if the effective temperature is lower (higher) than the environment temperature. According to Eq. (5), the ratio of the frequencies is constant in time, $\omega_j(t)/\omega_i(t) = \omega_j(0)/\omega_i(0)$, and it will be convenient to fix the ratio of adjacent frequencies as $\omega_j(0)/\omega_{j-1}(0) = e^\nu$, where ν determines their spacing. The initial oscillator frequencies are chosen so that some of them are always close to resonance with the system. The state in Eq. (3) at the initial time $t = 0$ can be produced by first fixing the frequencies to be $\omega_i(0)$ and then letting the harmonic oscillators equilibrate with the environment. Subsequently, the frequencies are changed to $\omega_i(0^+) = \omega_i(0)\mathcal{T}(0^+)/T_e$.

IV. TEMPERATURE PULSES

We now apply the effective temperature pulses indicated in brown in Fig. 1(b) with $\mathcal{T}(t) = T_0[1 + \lambda\cos(\Omega t)]$ to the left and a steplike profile to the right and then solve Eq. (2) with the time-dependent frequencies from Eq. (4) inserted. To this

end, we exploit the fact that Eq. (2) preserves the Gaussian nature of equilibrium states, which are fully described by their correlation matrix $\langle \hat{a}_k^\dagger \hat{a}_l \rangle$ with the equation of motion (see Appendix C)

$$\frac{d}{dt}\langle \hat{a}_k^\dagger \hat{a}_l \rangle = \sum_{m=0}^{N}(W_{km}\langle \hat{a}_m^\dagger \hat{a}_l \rangle + \langle \hat{a}_k^\dagger \hat{a}_m \rangle W_{ml}^*) + F_{kl}, \quad (6)$$

which follows from the Lindblad equation for the coupled quantum system and the harmonic oscillators [46]. Here, we have defined the $(N + 1) \times (N + 1)$ block matrix

$$\boldsymbol{W}(t) = i\begin{pmatrix} \omega_s & \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^\dagger & \boldsymbol{\Omega}(t) \end{pmatrix}, \quad (7)$$

whose first element contains the frequency of the system oscillator, and the row matrix $\boldsymbol{\alpha} = \alpha(1, 1, \ldots, 1)$ describes the coupling between the quantum system and the harmonic oscillators. We have also defined the diagonal matrix, $\Omega_{kl}(t) = \delta_{kl}(\omega_k(t) + i\gamma/2)$, with the N oscillator frequencies and the damping strength, while $\boldsymbol{F}$ is a diagonal matrix with 0 as the first element followed by $\gamma n_{T_e}(\omega_k)$ for each of the N oscillators. We note that $\langle \hat{a}_k^\dagger \rangle = \langle \hat{a}_k \rangle = 0$ for all oscillators. The system oscillator is always in a thermal state with the effective temperature given by the average occupation as $k_B T_s = \hbar\omega_s / \ln(1 + 1/\langle \hat{a}_s^\dagger \hat{a}_s \rangle) \simeq \hbar\omega_s \langle \hat{a}_s^\dagger \hat{a}_s \rangle$ for $\langle \hat{a}_s^\dagger \hat{a}_s \rangle \gg 1$ in agreement with the equipartition theorem.

Figure 1(b) shows the temperature of the quantum system in blue in response to the effective temperature pulses implemented with $N = 16$ harmonic oscillators. These results are based on solving Eq. (6) numerically. After a short transient, the temperature of the quantum system adapts to the periodic temperature drive and starts oscillating around the average effective temperature T_0. Since the driving is faster than the equilibration time of the system, the temperature oscillations are smaller than those of the effective temperature, and the phase lags behind the drive [20,47]. By contrast, if the drive was adiabatic, the system oscillator would all the time have the temperature of the harmonic oscillators.

To corroborate our results, we consider a Lindblad equation for the quantum system only reading

$$\frac{d\hat{\rho}_s(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_s, \hat{\rho}_s(t)] + \frac{2\pi}{\omega_s}\frac{\alpha^2}{\nu^2}\mathcal{D}_s(t)\hat{\rho}_s(t), \quad (8)$$

with the temperature drive $\mathcal{T}(t)$ entering via the Bose-Einstein distributions in the dissipator $\mathcal{D}_s(t)$. The prefactor $2\pi(\alpha/\nu)^2/\omega_s$ is related to the spacing of the oscillator frequencies, $\omega_j(0)/\omega_{j-1}(0) = e^\nu$, which ensures a roughly constant density of states around the frequency of the quantum system at all times. By contrast, the prefactor would be time dependent for other spacings of the frequencies. In Fig. 1(b), we show with black lines the system temperature calculated based on Eq. (8) and find good agreement with the solution of Eq. (6) in blue.

Figure 1 demonstrates that effective temperature pulses can be realized using $N = 16$ harmonic oscillators. It is, however, interesting to explore what happens if fewer oscillators are used. In Fig. 2, we show results obtained with $N = 4$, 8, and 12 oscillators together with the time-dependent frequencies for $N = 8$, which are chosen around the system frequency to ensure an efficient transfer of energy between the
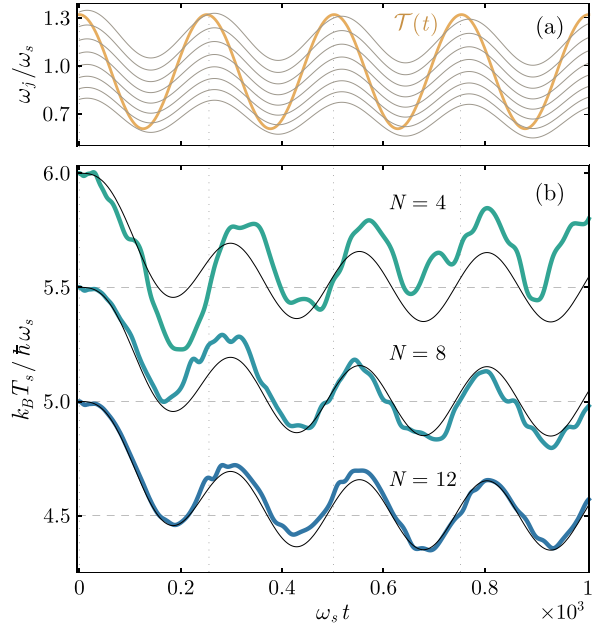

FIG. 2. System temperature for different numbers of oscillators. (a) We show the oscillator frequencies for $N = 8$ to realize the effective temperature pulses $\mathcal{T}(t)$ (plotted in arbitrary units). (b) The temperature of the quantum system depends on the number of harmonic oscillators, $N = 4, 8, 12$. For $N = 12$, it comes close to the time-dependent temperature obtained from a Lindblad equation for the quantum system only with $\mathcal{T}(t)$ inserted (black); see Eq. (8). The top and bottom curves have been displaced vertically, and we have used $\alpha = 0.05\nu\omega_s$, $\gamma = 0.01\omega_s$, and $k_B T_e = 5\hbar\omega_s$.

quantum system and the harmonic oscillators. Surprisingly, merely $N = 4$ oscillators are sufficient to mimic the effects of a time-dependent temperature, even if the quantum system might act back on the oscillators, causing them to deviate from the thermal state in Eq. (3). By contrast, with $N = 8$ and $N = 12$ oscillators, the agreement with the results of Eq. (8) is very good.

V. COOLING SCHEME

Our setup can also be used to cool the quantum system to a constant temperature below that of the environment, $\mathcal{T}(t) = T_0 < T_e$. To this end, we compress the frequencies as $\omega_j(t) = \omega_j(0)e^{(1-T_e/T_0)\gamma t}$ as illustrated in Fig. 3(a), and in the special case, where the oscillators are decoupled from the environment ($\gamma = 0$), we immediately quench the frequencies to $\omega_j(t) = \omega_j(0)T_0/T_e$ for $t > 0$, where T_e is the initial temperature. The resulting time-dependent temperature of the quantum system is shown in Fig. 3(b), where we see clear cooling effects as the temperature approaches the effective temperature of the oscillators. However, it is also obvious that the system cannot be kept cold indefinitely. To ensure an efficient removal of energy from the quantum system, it has to be on resonance with the oscillators, but eventually their frequencies are all reduced below the system frequency as seen in Fig. 3(a). Moreover, without a reservoir, N harmonic oscillators can only mimic an environment up to the time $t^* = 2(N - 1)^2/N\Delta\omega$, where $\Delta\omega$ is the difference between
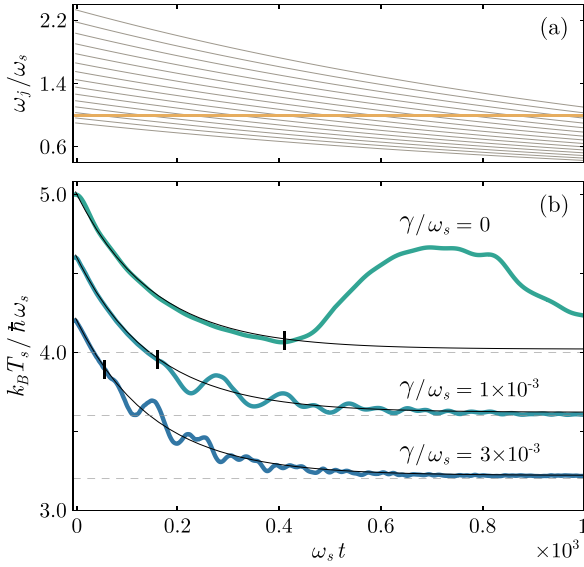
FIG. 3. Cooling scheme. (a) The oscillator frequencies are compressed to cool the system. (b) The system is cooled to the temperature $k_B T_0 = 4\hbar\omega_s$ below that of the environment, $k_B T_e = 5\hbar\omega_s$, using $N = 15$ oscillators and $\alpha = 0.05\nu\omega_s$ and $\gamma = 0, 0.001\omega_s, 0.003\omega_s$ for the three curves with the two lowest being displaced vertically. We also show the system temperature obtained from a Lindblad equation with the time-dependent temperature $\mathcal{T}(t)$ inserted (black); see Eq. (8). The markers indicate the times t^*, where the result are expected to deviate from the Lindblad equation (see text).

the largest and smallest oscillator frequencies [36], which we here evaluate at the initial time $\Delta\omega = \Delta\omega(0^+)$. These times are indicated in Fig. 3(b), and they coincide with the time at which we see deviations from the temperature based on Eq. (8). In particular, without the environment ($\gamma = 0$), the dynamics is reversible, and eventually heat flows back into the quantum system, causing it to heat up. This backflow is avoided by coupling the oscillators to the environment, as we observe for the two other curves with $\gamma > 0$, which eventually reach the desired temperature T_0.

Finally, we extend our cooling scheme by carrying out the cooling process in several steps as illustrated in Fig. 4. Figure 4(a) shows how we use each oscillator to cool the quantum system by first tuning the oscillator across the frequency of the quantum system, so that it cools down, and subsequently we increase the frequency so that it is away from resonance, and the oscillator is then reset by interacting with the environment before the next cooling step. By doing so periodically, we can cool the quantum system in a stepwise manner as shown in Fig. 4(b) for $N = 1, 3, 9$ oscillators. In particular, for $N = 9$, we observe a smooth cooling curve.

VI. EXPERIMENTAL PERSPECTIVES

Our scheme can be realized in a variety of experimental setups involving quantum harmonic oscillators, for example, mechanical nanoresonators [43–45,48,49], microwave cavities [50,51], or electromagnetic resonators [52,53]. With typical frequencies in the range $\omega_s \simeq 1$–10 GHz, the associated temperatures and timescales are about $T \simeq 50$–500 mK and $t \simeq 10^3/\omega_s \simeq 0.1$–1 μs, which are reachable in current
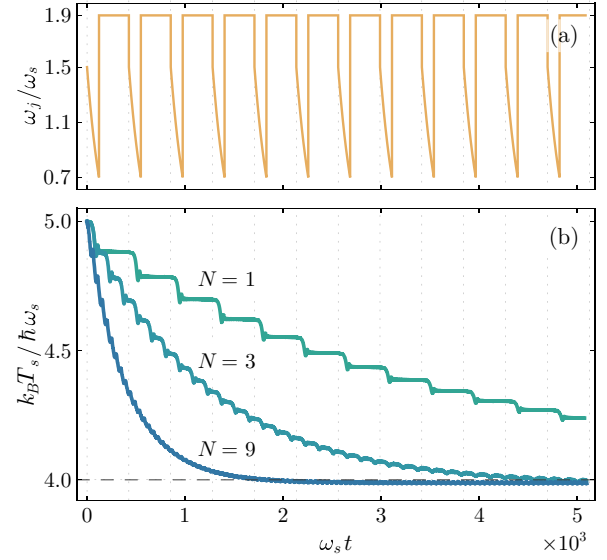

FIG. 4. Resetting protocol. (a) We show one of the frequencies in the resetting protocol. (b) Gradual cooling of the quantum system with $N = 1, 3, 9$ harmonic oscillators, and $\gamma = 0.025\omega_s$, $\alpha = 0.01\omega_s$, $k_B T_0 = 4\hbar\omega_s$, and $k_B T_e = 5\hbar\omega_s$.

low-temperature experiments. While we here have focused on quantum harmonic oscillators, we note that the scheme can also be implemented with fermionic two-level systems using Eq. (4) for the frequency splittings, however, with the Bose-Einstein distributions replaced by Fermi-Dirac distributions. Also, the open quantum system itself does not have to be a harmonic oscillator; rather, it can be any system with discrete energy levels.

VII. CONCLUSIONS AND OUTLOOK

We have proposed and analyzed a quantum mechanical scheme to control the effective temperature of an open quantum system. Specifically, by controlling the frequencies of a collection of harmonic oscillators that mediate the interactions between a quantum system and a thermal reservoir, we can implement arbitrary effective temperature drives that the quantum system experiences. We can thereby drive the quantum system with periodic temperature pulses, or it can be cooled to temperatures below that of the environment. Surprisingly, the scheme can be realized with only a few harmonic oscillators, and given the general nature of our proposal, we believe that it can be realized in a variety of experimental setups with the aim to perform coherent manipulations of small quantum systems. Our scheme may also be important for experiments that explore the foundations of quantum thermodynamics. Finally, our scheme may be extended in many directions, for example, to systems with memory effects [54].

APPENDIX A: DERIVATION AND VALIDITY OF EQUATION (2)

Here, we derive Eq. (2), inspired by the derivation in Ref. [41] for a constant Hamiltonian. To extend the discussion to a time-dependent Hamiltonian, we follow the derivation in Ref. [42], which we adapt to our specific setup. In doing so, we identify the particular conditions for Eq. (2) to hold, which we indicate in boxes below.

Our starting point is the total Hamiltonian

$$\hat{H}_T(t) = \hat{H}(t) + \hat{H}_B + \hat{H}_I, \tag{A1}$$

which describes several coupled oscillators. The time-dependent term is given by Eq. (1) and reads

$$\hat{H}(t) = \hbar\omega_s(\hat{a}_s^\dagger\hat{a}_s + 1/2) + \sum_{j=1}^{N} \hbar\omega_j(t)(\hat{a}_j^\dagger\hat{a}_j + 1/2)$$

$$+ \hbar\alpha \sum_{j=1}^{N}(\hat{a}_s^\dagger\hat{a}_j + \hat{a}_j^\dagger\hat{a}_s), \tag{A2}$$

with $\hat{a}_{s,j}^{(\dagger)}$ being the usual ladder operators. The first oscillator (with frequency ω_s) is the system oscillator, while we refer to the others [with frequencies $\omega_j(t)$] as the ancilla oscillators. The coupling between them is denoted by α. We model the thermal bath with a Hamiltonian that describes a continuous spectrum of harmonic oscillators,

$$\hat{H}_B = \hbar \int_0^{\omega_m} d\omega\, \hat{b}_\omega^\dagger \hat{b}_\omega, \tag{A3}$$

where $\hat{b}_\omega^{(\dagger)}$ are the ladder operators of the bath oscillators and ω_m is a cutoff frequency. Finally, the interaction between the ancilla oscillators and the bath is described by the interaction Hamiltonian

$$\hat{H}_I = \hbar g \sum_{j=1}^{N} \int_0^{\omega_m} d\omega\, h_j[\omega](\hat{a}_j^\dagger \hat{b}_\omega + \hat{a}_j \hat{b}_\omega^\dagger), \tag{A4}$$

where g is the interaction strength and $h_j[\omega]$ is a function that describes the coupling to each oscillator.

The full system-ancilla-bath density matrix evolves according to the Liouville–von Neumann equation

$$\frac{d}{dt}\hat{\rho}_T(t) = -\frac{i}{\hbar}[\hat{H}_T(t), \hat{\rho}_T(t)]. \tag{A5}$$

Our aim is to describe the system-ancilla dynamics itself, taking into account the coupling to the thermal bath. To this end, we switch to the interaction picture with respect to $\hat{H}(t) + \hat{H}_B$

by introducing the unitary operator

$$\hat{U}_0(t, t_0) = \hat{T}\left\{e^{-i\int_{t_0}^{t} ds[\hat{H}(s) + \hat{H}_B]/\hbar}\right\}$$

$$= \hat{T}\left\{e^{-i\int_{t_0}^{t} ds\hat{H}(s)/\hbar}\right\} \otimes e^{-i\hat{H}_B(t-t_0)/\hbar}$$

$$\equiv \hat{U}(t, t_0) \otimes \hat{U}_B(t, t_0), \tag{A6}$$

where $\hat{T}$ is the chronological time-ordering operator. In the interaction picture, the full density matrix is defined as $\tilde{\rho}_T(t) = \hat{U}_0^\dagger(t, 0)\hat{\rho}_T(t)\hat{U}_0(t, 0)$, and its equation of motion easily follows as

$$\frac{d}{dt}\tilde{\rho}_T(t) = -\frac{i}{\hbar}[\tilde{H}_I(t), \tilde{\rho}_T(t)], \tag{A7}$$

with the interaction Hamiltonian in the interaction picture given by $\tilde{H}_I(t) = \hat{U}_0^\dagger(t, 0)\hat{H}_I\hat{U}_0(t, 0)$. Equation (A7) can be integrated as $\tilde{\rho}_T(t) = \tilde{\rho}_T(0) - \frac{i}{\hbar}\int_0^t ds[\tilde{H}_I(s), \tilde{\rho}_T(s)]$ and iterated once to yield

$$\tilde{\rho}_T(t) = \tilde{\rho}_T(0) - \frac{i}{\hbar}\int_0^t ds[\tilde{H}_I(s), \tilde{\rho}_T(0)]$$

$$- \frac{1}{\hbar^2}\int_0^t ds \int_0^s ds'[\tilde{H}_I(s), [\tilde{H}_I(s'), \tilde{\rho}_T(s')]]. \tag{A8}$$

Differentiating this expression with respect to time, we find

$$\frac{d}{dt}\tilde{\rho}_T(t) = -\frac{i}{\hbar}[\tilde{H}_I(t), \tilde{\rho}_T(0)]$$

$$- \frac{1}{\hbar^2}\left[\tilde{H}_I(t), \int_0^t ds[\tilde{H}_I(t-s), \tilde{\rho}_T(t-s)]\right]. \tag{A9}$$

Next, we consider the system-ancilla density matrix by tracing out the bath degrees of freedom, yielding

$$\tilde{\rho}(t) = \mathrm{tr}_B\{\hat{U}_0^\dagger(t, 0)\hat{\rho}_T(t)\hat{U}_0(t, 0)\}$$

$$= \hat{U}^\dagger(t, 0)\mathrm{tr}_B\{\hat{U}_B^\dagger(t, 0)\hat{\rho}_T(t)\hat{U}_B(t, 0)\}\hat{U}(t, 0)$$

$$= \hat{U}^\dagger(t, 0)\hat{\rho}(t)\hat{U}(t, 0), \tag{A10}$$

where we have used the fact that $\mathrm{tr}_B\{\hat{U}_B^\dagger(t, 0)\hat{\rho}_T(t)\hat{U}_B(t, 0)\} = \mathrm{tr}_B\{\hat{U}_B(t, 0)\hat{U}_B^\dagger(t, 0)\hat{\rho}_T(t)\} = \hat{\rho}(t)$. From Eq. (A9), we then find

$$\frac{d}{dt}\tilde{\rho}(t) = -\frac{i}{\hbar}\mathrm{tr}_B\{[\tilde{H}_I(t), \tilde{\rho}_T(0)]\}$$

$$- \frac{1}{\hbar^2}\mathrm{tr}_B\left\{\left[\tilde{H}_I(t), \int_0^t ds[\tilde{H}_I(t-s), \tilde{\rho}_T(t-s)]\right]\right\}. \tag{A11}$$

Up until this point, we have not made any approximations. However, to proceed, we first apply the standard Born approximation, where correlations between the system and the ancilla, on the one hand, and the bath, on the other hand, are neglected, and we may factorize the density matrix as

(i) $\tilde{\rho}_T(t) = \tilde{\rho}(t) \otimes \tilde{\rho}_B(t)$, with $\tilde{\rho}_B(t) = \hat{\rho}_B = e^{-\hat{H}_B/(k_B T_e)}/\mathrm{tr}\{e^{-\hat{H}_B/(k_B T_e)}\}$ (Born approximation).

We note that the Born approximation in the context of time-dependent problems is discussed in more detail in Ref. [42]. We have also assumed that the bath remains in its thermal state with a fixed temperature T_e. Furthermore, we note that the first term in Eq. (A11) vanishes since

$$\mathrm{tr}_B\{[\tilde{H}_I(t), \tilde{\rho}_T(0)]\} = \mathrm{tr}_B\{[\tilde{H}_I(t), \tilde{\rho}(0) \otimes \hat{\rho}_B]\} = 0 \tag{A12}$$

for the thermal bath state. Together with the explicit expression for the interaction Hamiltonian

$$\tilde{H}_I(t) = \hat{U}_0^\dagger(t,0)\hat{H}_I\hat{U}_0(t,0) = \hbar g \sum_{j=1}^{N} \int_0^{\omega_m} d\omega h_j[\omega][\tilde{a}_j^\dagger(t)\tilde{b}_\omega(t) + \tilde{a}_j(t)\tilde{b}_\omega^\dagger(t)], \tag{A13}$$

where $\tilde{a}_j^{(\dagger)}(t) = \hat{U}^\dagger(t,0)\hat{a}_j^{(\dagger)}\hat{U}(t,0)$ and $\tilde{b}_\omega^{(\dagger)}(t) = \hat{U}_B^\dagger(t,0)\hat{b}_\omega^{(\dagger)}\hat{U}_B(t,0)$ are the ladder operators in the interaction picture, we then obtain from Eq. (A11) the expression

$$\frac{d}{dt}\tilde{\rho}(t) = -g^2 \sum_{j=1}^{N} \sum_{k=1}^{N} \int_0^{\omega_m} d\omega h_j[\omega] \int_0^{\omega_m} d\omega' h_k[\omega'] \int_0^t ds$$

$$\times \text{tr}_B\{[\tilde{a}_j^\dagger(t)\tilde{b}_\omega(t) + \tilde{a}_j(t)\tilde{b}_\omega^\dagger(t), [\tilde{a}_k^\dagger(t-s)\tilde{b}_{\omega'}(t-s) + \tilde{a}_k(t-s)\tilde{b}_{\omega'}^\dagger(t-s), \tilde{\rho}(t-s) \otimes \hat{\rho}_B]]\}. \tag{A14}$$

Writing out the commutators, we find

$$\frac{d}{dt}\tilde{\rho}(t) = g^2 \sum_{j=1}^{N} \sum_{k=1}^{N} \int_0^{\omega_m} d\omega h_j[\omega] \int_0^{\omega_m} d\omega' h_k[\omega'] \int_0^t ds([\tilde{a}_k^\dagger(t-s)\tilde{\rho}(t-s)\tilde{a}_j^\dagger(t) - \tilde{a}_j^\dagger(t)\tilde{a}_k^\dagger(t-s)\tilde{\rho}(t-s)]\langle\tilde{b}_\omega(t)\tilde{b}_{\omega'}(t-s)\rangle$$

$$+ [\tilde{a}_j^\dagger(t)\tilde{\rho}(t-s)\tilde{a}_k^\dagger(t-s) - \tilde{\rho}(t-s)\tilde{a}_k^\dagger(t-s)\tilde{a}_j^\dagger(t)]\langle\tilde{b}_{\omega'}(t-s)\tilde{b}_\omega(t)\rangle + [\tilde{a}_k(t-s)\tilde{\rho}(t-s)\tilde{a}_j^\dagger(t)$$

$$- \tilde{a}_j^\dagger(t)\tilde{a}_k(t-s)\tilde{\rho}(t-s)]\langle\tilde{b}_\omega(t)\tilde{b}_{\omega'}^\dagger(t-s)\rangle + [\tilde{a}_j^\dagger(t)\tilde{\rho}(t-s)\tilde{a}_k(t-s) - \tilde{\rho}(t-s)\tilde{a}_k(t-s)\tilde{a}_j^\dagger(t)]\langle\tilde{b}_{\omega'}^\dagger(t-s)\tilde{b}_\omega(t)\rangle$$

$$+ [\tilde{a}_k^\dagger(t-s)\tilde{\rho}(t-s)\tilde{a}_j(t) - \tilde{a}_j(t)\tilde{a}_k^\dagger(t-s)\tilde{\rho}(t-s)]\langle\tilde{b}_\omega^\dagger(t)\tilde{b}_{\omega'}(t-s)\rangle + [\tilde{a}_j(t)\tilde{\rho}(t-s)\tilde{a}_k^\dagger(t-s)$$

$$- \tilde{\rho}(t-s)\tilde{a}_k^\dagger(t-s)\tilde{a}_j(t)]\langle\tilde{b}_{\omega'}(t-s)\tilde{b}_\omega^\dagger(t)\rangle + [\tilde{a}_k(t-s)\tilde{\rho}(t-s)\tilde{a}_j(t) - \tilde{a}_j(t)\tilde{a}_k(t-s)\tilde{\rho}(t-s)]\langle\tilde{b}_\omega^\dagger(t)\tilde{b}_{\omega'}^\dagger(t-s)\rangle$$

$$+ [\tilde{a}_j(t)\tilde{\rho}(t-s)\tilde{a}_k(t-s) - \tilde{\rho}(t-s)\tilde{a}_k(t-s)\tilde{a}_j(t)]\langle\tilde{b}_{\omega'}^\dagger(t-s)\tilde{b}_\omega^\dagger(t)\rangle), \tag{A15}$$

where all expectation values are evaluated with respect to the equilibrium state of the bath.

Since the state of the bath is assumed to be stationary, the bath correlation functions are translational invariant in time, $\langle\tilde{b}_\omega^{(\dagger)}(t)\tilde{b}_{\omega'}^{(\dagger)}(t-s)\rangle = \langle\tilde{b}_\omega^{(\dagger)}(s)\tilde{b}_{\omega'}^{(\dagger)}(0)\rangle$. In the following, we let τ_B denote the characteristic correlation time scale on which the bath correlations $\langle\tilde{b}_\omega^{(\dagger)}(t)\tilde{b}_{\omega'}^{(\dagger)}(0)\rangle \sim e^{-t/\tau_B}$ decay. Considering sufficiently long times, $t \gg \tau_B$, and assuming that the system-ancilla relaxation time τ_R (which in the interaction picture only depends on $g \ll 1$ and not on the internal time scales of the system and the ancilla) is much longer than the relaxation dynamics of the bath, $\tau_R \gg \tau_B$, we can take

$$\boxed{\text{(ii)} \quad \tilde{\rho}(t-s) \simeq \tilde{\rho}(t) \quad \textit{(Markov approximation)}}$$

and extend the upper integration limit to infinity. We then obtain

$$\frac{d}{dt}\tilde{\rho}(t) = g^2 \sum_{j=1}^{N} \sum_{k=1}^{N} \int_0^{\omega_m} d\omega h_j[\omega] \int_0^{\omega_m} d\omega' h_k[\omega'] \int_0^\infty ds([\tilde{a}_k^\dagger(t-s)\tilde{\rho}(t)\tilde{a}_j^\dagger(t) - \tilde{a}_j^\dagger(t)\tilde{a}_k^\dagger(t-s)\tilde{\rho}(t)]\langle\tilde{b}_\omega(s)\tilde{b}_{\omega'}(0)\rangle$$

$$+ [\tilde{a}_j^\dagger(t)\tilde{\rho}(t)\tilde{a}_k^\dagger(t-s) - \tilde{\rho}(t)\tilde{a}_k^\dagger(t-s)\tilde{a}_j^\dagger(t)]\langle\tilde{b}_{\omega'}(0)\tilde{b}_\omega(s)\rangle + [\tilde{a}_k(t-s)\tilde{\rho}(t)\tilde{a}_j^\dagger(t) - \tilde{a}_j^\dagger(t)\tilde{a}_k(t-s)\tilde{\rho}(t)]\langle\tilde{b}_\omega(s)\tilde{b}_{\omega'}^\dagger(0)\rangle$$

$$+ [\tilde{a}_j^\dagger(t)\tilde{\rho}(t)\tilde{a}_k(t-s) - \tilde{\rho}(t)\tilde{a}_k(t-s)\tilde{a}_j^\dagger(t)]\langle\tilde{b}_{\omega'}^\dagger(0)\tilde{b}_\omega(s)\rangle + [\tilde{a}_k^\dagger(t-s)\tilde{\rho}(t)\tilde{a}_j(t) - \tilde{a}_j(t)\tilde{a}_k^\dagger(t-s)\tilde{\rho}(t)]\langle\tilde{b}_\omega^\dagger(s)\tilde{b}_{\omega'}(0)\rangle$$

$$+ [\tilde{a}_j(t)\tilde{\rho}(t)\tilde{a}_k^\dagger(t-s) - \tilde{\rho}(t)\tilde{a}_k^\dagger(t-s)\tilde{a}_j(t)]\langle\tilde{b}_{\omega'}(0)\tilde{b}_\omega^\dagger(s)\rangle + [\tilde{a}_k(t-s)\tilde{\rho}(t)\tilde{a}_j(t) - \tilde{a}_j(t)\tilde{a}_k(t-s)\tilde{\rho}(t)]\langle\tilde{b}_\omega^\dagger(s)\tilde{b}_{\omega'}^\dagger(0)\rangle$$

$$+ [\tilde{a}_j(t)\tilde{\rho}(t)\tilde{a}_k(t-s) - \tilde{\rho}(t)\tilde{a}_k(t-s)\tilde{a}_j(t)]\langle\tilde{b}_{\omega'}^\dagger(0)\tilde{b}_\omega^\dagger(s)\rangle). \tag{A16}$$

We now express the ladder operators in the interaction picture in terms of the operators in the Schrödinger picture. To this end, we consider the regime of a weak coupling between the system and the ancilla,

$$\boxed{\text{(iii)} \quad \alpha \ll \omega_s, \omega_j \quad \textit{(system} - \textit{ancilla weak coupling regime)},}$$

such that the ladder operators $\hat{a}_j^{(\dagger)}$ are (approximate) eigenoperators of the superoperator $\mathcal{H}(t) = [\hat{H}(t), \cdot]$, with $\mathcal{H}(t)\hat{a}_j^\dagger = \hbar\omega_j(t)\hat{a}_j^\dagger$ and $\mathcal{H}(t)\hat{a}_j = -\hbar\omega_j(t)\hat{a}_j$. We note that condition (iii) is particular to our model, and we work with this condition since the quantum system should only be weakly coupled to its environment. We then obtain

$$\tilde{a}_j(t) = \hat{U}^\dagger(t,0)\hat{a}_j\hat{U}(t,0) = \hat{T}^*\{e^{i\int_0^t ds'\hat{H}(s')/\hbar}\}\hat{a}_j\hat{T}\{e^{-i\int_0^t ds'\hat{H}(s')/\hbar}\} = e^{-i\int_0^t ds'\omega_j(s')}\hat{a}_j,$$

$$\tilde{a}_j^\dagger(t) = \hat{U}^\dagger(t,0)\hat{a}_j^\dagger\hat{U}(t,0) = \hat{T}^*\{e^{i\int_0^t ds'\hat{H}(s')/\hbar}\}\hat{a}_j^\dagger\hat{T}\{e^{-i\int_0^t ds'\hat{H}(s')/\hbar}\} = e^{i\int_0^t ds'\omega_j(s')}\hat{a}_j^\dagger, \tag{A17}$$

where $\hat{T}^*$ is the antichronological time-ordering operator. Furthermore, for $\tilde{a}_k^{(\dagger)}(t-s)$, we assume that $\hat{H}(t)$ changes on a time scale τ_A that is much longer than the time scale τ_B on which the correlation functions $\langle\tilde{b}_\omega^{(\dagger)}(t)\tilde{b}_{\omega'}^{(\dagger)}(0)\rangle$ decay,

$$\boxed{\text{(iv)} \quad \tau_A \gg \tau_B \quad \textit{(slow driving compared with the decay of bath correlations)}.}$$

Importantly, this allows us to make the approximation

$$\hat{U}(t-s,0) = \hat{U}^{\dagger}(t,t-s)\hat{U}(t,0) \simeq e^{i\hat{H}(t)s/\hbar}\hat{U}(t,0), \tag{A18}$$

where we have used the fact that $\hat{H}(t)$ is approximately constant on the time scale on which the bath correlations decay, $\hat{U}^{\dagger}(t,t-s) = \hat{T}^*\{e^{i\int_{t-s}^{t} ds'\hat{H}(s')/\hbar}\} \simeq e^{i\hat{H}(t)\int_{t-s}^{t} ds'/\hbar} = e^{i\hat{H}(t)s/\hbar}$. Using this approximation, we obtain

$$\tilde{a}_k(t-s) = \hat{U}^{\dagger}(t-s,0)\hat{a}_k\hat{U}(t-s,0) \simeq \hat{T}^*\{e^{i\int_0^t ds'\hat{H}(s')/\hbar}\}e^{-i\hat{H}(t)s/\hbar}a_k e^{i\hat{H}(t)s/\hbar}\hat{T}\{e^{-i\int_0^t ds'\hat{H}(s')/\hbar}\} = e^{-i\int_0^t ds'\omega_k(s')}e^{is\omega_k(t)}a_k,$$

$$\tilde{a}_k^{\dagger}(t-s) = \hat{U}^{\dagger}(t-s,0)\hat{a}_k^{\dagger}\hat{U}(t-s,0) \simeq \hat{T}^*\{e^{i\int_0^t ds'\hat{H}(s')/\hbar}\}e^{-i\hat{H}(t)s/\hbar}a_k^{\dagger} e^{i\hat{H}(t)s/\hbar}\hat{T}\{e^{-i\int_0^t ds'\hat{H}(s')/\hbar}\} = e^{i\int_0^t ds'\omega_k(s')}e^{-is\omega_k(t)}a_k^{\dagger}. \tag{A19}$$

Plugging the equations above back into Eq. (A16), we obtain

$$\frac{d}{dt}\tilde{\rho}(t) = g^2 \sum_{j=1}^{N}\sum_{k=1}^{N}\int_0^{\omega_m} d\omega h_j[\omega]\int_0^{\omega_m} d\omega' h_k[\omega']\int_0^{\infty} ds$$

$$\times [([\hat{a}_k^{\dagger}\tilde{\rho}(t)\hat{a}_j^{\dagger} - \hat{a}_j^{\dagger}\hat{a}_k^{\dagger}\tilde{\rho}(t)]\langle\tilde{b}_{\omega}(s)\tilde{b}_{\omega'}(0)\rangle + [\hat{a}_j^{\dagger}\tilde{\rho}(t)\hat{a}_k^{\dagger} - \tilde{\rho}(t)\hat{a}_k^{\dagger}\hat{a}_j^{\dagger}]\langle\tilde{b}_{\omega'}(0)\tilde{b}_{\omega}(s)\rangle)e^{-is\omega_k(t)+i\int_0^t ds'[\omega_j(s')+\omega_k(s')]}$$

$$+ ([\hat{a}_k\tilde{\rho}(t)\hat{a}_j^{\dagger} - \hat{a}_j^{\dagger}\hat{a}_k\tilde{\rho}(t)]\langle\tilde{b}_{\omega}(s)\tilde{b}_{\omega'}^{\dagger}(0)\rangle + [\hat{a}_j^{\dagger}\tilde{\rho}(t)\hat{a}_k - \tilde{\rho}(t)\hat{a}_k\hat{a}_j^{\dagger}]\langle\tilde{b}_{\omega'}^{\dagger}(0)\tilde{b}_{\omega}(s)\rangle)e^{is\omega_k(t)+i\int_0^t ds'[\omega_j(s')-\omega_k(s')]}$$

$$+ ([\hat{a}_k^{\dagger}\tilde{\rho}(t)\hat{a}_j - \hat{a}_j\hat{a}_k^{\dagger}\tilde{\rho}(t)]\langle\tilde{b}_{\omega}^{\dagger}(s)\tilde{b}_{\omega'}(0)\rangle + [\hat{a}_j\tilde{\rho}(t)\hat{a}_k^{\dagger} - \tilde{\rho}(t)\hat{a}_k^{\dagger}\hat{a}_j]\langle\tilde{b}_{\omega'}(0)\tilde{b}_{\omega}^{\dagger}(s)\rangle)e^{-is\omega_k(t)-i\int_0^t ds'[\omega_j(s')-\omega_k(s')]}$$

$$+ ([\hat{a}_k\tilde{\rho}(t)\hat{a}_j - \hat{a}_j\hat{a}_k\tilde{\rho}(t)]\langle\tilde{b}_{\omega}^{\dagger}(s)\tilde{b}_{\omega'}^{\dagger}(0)\rangle + [\hat{a}_j\tilde{\rho}(t)\hat{a}_k - \tilde{\rho}(t)\hat{a}_k\hat{a}_j]\langle\tilde{b}_{\omega'}^{\dagger}(0)\tilde{b}_{\omega}^{\dagger}(s)\rangle)e^{is\omega_k(t)-i\int_0^t ds'[\omega_j(s')+\omega_k(s')]}]. \tag{A20}$$

We now perform the secular approximation. To this end, we note that for our driving protocols, $\omega_j(t) \neq \omega_k(t)$ for all times, unless $j = k$. Under the assumption that the inverse frequency differences, for all times, are much smaller than the system-ancilla relaxation time,

$$\boxed{\text{(v)} \quad |\omega_j(t) - \omega_k(t)|^{-1} \ll \tau_R \quad \textit{(condition for the secular approximation)},}$$

we may neglect all terms except those with a vanishing integral in the exponent, as they oscillate much faster than the typical time scale of the system-ancilla dynamics. This approximation then yields

$$\frac{d}{dt}\tilde{\rho}(t) = g^2 \sum_{j=1}^{N}\int_0^{\omega_m} d\omega h_j[\omega]\int_0^{\omega_m} d\omega' h_j[\omega']\int_0^{\infty} ds$$

$$\times ([\hat{a}_j\tilde{\rho}(t)\hat{a}_j^{\dagger} - \hat{a}_j^{\dagger}\hat{a}_j\tilde{\rho}(t)]\langle\tilde{b}_{\omega}(s)\tilde{b}_{\omega'}^{\dagger}(0)\rangle e^{is\omega_j(t)} + [\hat{a}_j^{\dagger}\tilde{\rho}(t)\hat{a}_j - \tilde{\rho}(t)\hat{a}_j\hat{a}_j^{\dagger}]\langle\tilde{b}_{\omega'}^{\dagger}(0)\tilde{b}_{\omega}(s)\rangle e^{is\omega_j(t)}$$

$$+ [\hat{a}_j^{\dagger}\tilde{\rho}(t)\hat{a}_j - \hat{a}_j\hat{a}_j^{\dagger}\tilde{\rho}(t)]\langle\tilde{b}_{\omega}^{\dagger}(s)\tilde{b}_{\omega'}(0)\rangle e^{-is\omega_j(t)} + [\hat{a}_j\tilde{\rho}(t)\hat{a}_j^{\dagger} - \tilde{\rho}(t)\hat{a}_j^{\dagger}\hat{a}_j]\langle\tilde{b}_{\omega'}(0)\tilde{b}_{\omega}^{\dagger}(s)\rangle e^{-is\omega_j(t)}). \tag{A21}$$

Introducing the bath correlation functions in the frequency domain as

$$\Gamma_{\omega\omega'}[\omega_j(t)] = \int_0^{\infty} ds\langle\tilde{b}_{\omega}(s)\tilde{b}_{\omega'}(0)\rangle e^{-is\omega_j(t)} \equiv \frac{1}{2}\mu_{\omega\omega'}[\omega_j(t)] + i\eta_{\omega\omega'}[\omega_j(t)], \tag{A22}$$

and the notation $\tilde{b}_{-\omega}(t) \equiv \tilde{b}_{\omega}^{\dagger}(t)$, we obtain

$$\frac{d}{dt}\tilde{\rho}(t) = g^2 \sum_{j=1}^{N}\int_0^{\omega_m} d\omega h_j[\omega]\int_0^{\omega_m} d\omega' h_j[\omega']([\hat{a}_j\tilde{\rho}(t)\hat{a}_j^{\dagger} - \hat{a}_j^{\dagger}\hat{a}_j\tilde{\rho}(t)]\Gamma_{\omega-\omega'}[-\omega_j(t)] + [\hat{a}_j\tilde{\rho}(t)\hat{a}_j^{\dagger} - \tilde{\rho}(t)\hat{a}_j^{\dagger}\hat{a}_j]\Gamma_{-\omega'\omega}^{*}[-\omega_j(t)]$$

$$+ [\hat{a}_j^{\dagger}\tilde{\rho}(t)\hat{a}_j - \hat{a}_j\hat{a}_j^{\dagger}\tilde{\rho}(t)]\Gamma_{-\omega\omega'}[\omega_j(t)] + [\hat{a}_j^{\dagger}\tilde{\rho}(t)\hat{a}_j - \tilde{\rho}(t)\hat{a}_j\hat{a}_j^{\dagger}]\Gamma_{\omega'-\omega}^{*}[\omega_j(t)]). \tag{A23}$$

Separating the Hermitian and non-Hermitian parts of the bath correlation functions, and then transforming back to the Schrödinger picture, we obtain

$$\frac{d}{dt}\hat{\rho}(t) = -\frac{i}{\hbar}[\hat{H}(t) + \hat{H}_{Ls}(t), \hat{\rho}(t)]$$

$$+ \sum_{j=1}^{N}\left[\gamma_j^{-}[\omega_j(t)]\left(\hat{a}_j\hat{\rho}(t)\hat{a}_j^{\dagger} - \frac{1}{2}\{\hat{a}_j^{\dagger}\hat{a}_j, \hat{\rho}(t)\}\right) + \gamma_j^{+}[\omega_j(t)]\left(\hat{a}_j^{\dagger}\hat{\rho}(t)\hat{a}_j - \frac{1}{2}\{\hat{a}_j\hat{a}_j^{\dagger}, \hat{\rho}(t)\}\right)\right], \tag{A24}$$

where

$$\gamma_j^{\pm}[\omega_j(t)] = g^2 \int_0^{\omega_m} d\omega h_j[\omega]\int_0^{\omega_m} d\omega' h_j[\omega']\mu_{\mp\omega\pm\omega'}[\pm\omega_j(t)] \tag{A25}$$

are the absorption rates ($\gamma_j^+[\omega]$) and the emission rates ($\gamma_j^-[\omega]$) of quanta from/to the thermal bath, and

$$\hat{H}_{Ls}(t) = g^2 \sum_{j=1}^{N} \int_0^{\omega_m} d\omega h_j[\omega] \int_0^{\omega_m} d\omega' h_j[\omega'] (\eta_{\omega-\omega'}[-\omega_j(t)]\hat{a}_j^\dagger \hat{a}_j + \eta_{-\omega\omega'}[\omega_j(t)]\hat{a}_j \hat{a}_j^\dagger) \tag{A26}$$

is the Lamb-shift Hamiltonian, which commutes with $\hat{H}(t)$ for $\alpha \ll \omega_s, \omega_j$, and thus only leads to a renormalization of the energy levels. We note that this renormalization is proportional to $g^2 \ll 1$, and thus we neglect it in the following. In addition, we can derive an explicit expression for the emission rate of quanta to the bath

$$\gamma_j^-[\omega_j(t)] = g^2 \int_0^{\omega_m} d\omega h_j[\omega] \int_0^{\omega_m} d\omega' h_j[\omega'] \mu_{\omega-\omega'}[-\omega_j(t)]$$

$$= g^2 \int_0^{\omega_m} d\omega h_j[\omega] \int_0^{\omega_m} d\omega' h_j[\omega'] \int_{-\infty}^{\infty} ds e^{is\omega_j(t)} \langle \tilde{b}_\omega(s) \tilde{b}_{\omega'}^\dagger(0) \rangle, \tag{A27}$$

in terms of the spectral density. Using the fact that $\tilde{b}_\omega(t) = \hat{U}_B^\dagger(t,0)\hat{b}_\omega \hat{U}_B(t,0) = e^{-i\omega s}\hat{b}_\omega$, we obtain

$$\gamma_j^-[\omega_j(t)] = g^2 \int_0^{\omega_m} d\omega h_j[\omega] \int_0^{\omega_m} d\omega' h_j[\omega'] \int_{-\infty}^{\infty} ds e^{is[\omega_j(t)-\omega]} \langle \hat{b}_\omega \hat{b}_{\omega'}^\dagger \rangle = 2\pi g^2 \int_0^{\omega_m} d\omega' h_j[\omega'] h_j[\omega_j(t)] \langle \hat{b}_{\omega_j(t)} \hat{b}_{\omega'}^\dagger \rangle$$

$$= 2\pi g^2 h_j^2[\omega_j(t)] \langle \hat{b}_{\omega_j(t)} \hat{b}_{\omega_j(t)}^\dagger \rangle \equiv 2\pi g^2 h_j^2[\omega_j(t)](1 + n_{T_e}[\omega_j(t)]), \tag{A28}$$

provided that $0 < \omega_j(t) < \omega_m$. Here, $n_{T_e}[\omega]$ is the Bose-Einstein distribution, and we have used the fact that $\int_{-\infty}^{\infty} ds e^{is\omega} = 2\pi\delta(\omega)$, with $\delta(\omega)$ being the Dirac delta function. Introducing the spectral density $J_j[\omega] \equiv 2\pi h_j^2[\omega]$, we find

$$\gamma_j^-[\omega_j(t)] = g^2 J_j[\omega_j(t)](1 + n_{T_e}[\omega_j(t)]). \tag{A29}$$

Analogously, we find a similar expression for the absorption rate of quanta from the bath reading

$$\gamma_j^+[\omega_j(t)] = g^2 J_j[\omega_j(t)] n_{T_e}[\omega_j(t)]. \tag{A30}$$

We note that the rates fulfill detailed balance, $\gamma_j^+[\omega_j(t)]/\gamma_j^-[\omega_j(t)] = n_{T_e}[\omega_j(t)]/(1 + n_{T_e}[\omega_j(t)]) = e^{-\hbar\omega/(k_B T_e)}$ as expected. Using the expressions for the rates, we finally arrive at the master equation in Eq. (2) of the main text,

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}[\hat{H}(t), \hat{\rho}(t)] + \sum_{j=1}^{N} \gamma_j[\omega_j(t)]\mathcal{D}_j(t)\hat{\rho}(t), \tag{A31}$$

with $\gamma_j[\omega_j(t)] = g^2 J_j[\omega_j(t)]$ and the dissipator

$$\mathcal{D}_j\hat{\rho} = (1 + n_{T_e}[\omega_j(t)])(\hat{a}_j\hat{\rho}\hat{a}_j^\dagger - \tfrac{1}{2}\{\hat{a}_j^\dagger\hat{a}, \hat{\rho}\})$$
$$+ n_{T_e}[\omega_j(t)](\hat{a}_j^\dagger\hat{\rho}\hat{a}_j - \tfrac{1}{2}\{\hat{a}_j\hat{a}_j^\dagger, \hat{\rho}\}). \tag{A32}$$

APPENDIX B: DERIVATION OF EQUATION (4)

As our starting point, we take Eq. (2) with $\alpha = 0$, for which it for the oscillators simplifies to

$$\frac{d\hat{\rho}(t)}{dt} = -i \sum_{j=1}^{N} \omega_j[\hat{a}_j^\dagger\hat{a}_j, \hat{\rho}(t)] + \sum_{j=1}^{N} \gamma_j(\omega_j)\mathcal{D}_j\hat{\rho}(t), \tag{B1}$$

having omitted the explicit time dependence of the oscillator frequencies, $\omega_j = \omega_j(t)$. In this case, the harmonic oscillators

are independent, and we can treat each of them separately as

$$\frac{d\hat{\rho}_j(t)}{dt} = -i\omega_j[\hat{a}_j^\dagger\hat{a}_j, \hat{\rho}_j(t)] + \gamma_j(\omega_j)\mathcal{D}_j\hat{\rho}_j(t), \quad j = 1, \ldots, N. \tag{B2}$$

Next, we introduce a characteristic function for each oscillator by defining

$$\chi_j(\lambda, t) = \mathrm{Tr}(\hat{\rho}_j(t)\hat{D}_j(\lambda)), \quad \hat{D}_j(\lambda) = e^{\lambda\hat{a}_j^\dagger - \lambda^*\hat{a}_j}, \tag{B3}$$

where $\hat{D}_j(\lambda)$ is the displacement operator. The equation of motion for $\chi_j(\lambda, t)$ follows from Eq. (B2) and reads

$$\frac{d\chi_j(\lambda, t)}{dt} = \mathcal{U}_j\chi_j(\lambda, t) + \mathcal{D}_j\chi_j(\lambda, t), \tag{B4}$$

where

$$\mathcal{U}_j\chi_j(\lambda, t) = -i\omega_j(t)[\lambda^*\partial_{\lambda^*} - \lambda\partial_\lambda]\chi_j(\lambda, t) \tag{B5}$$

corresponds to the unitary dynamics, while the dissipator leads to the term

$$\mathcal{D}_j\chi_j(\lambda, t) = -(\gamma(\omega_j)/2)[\lambda^*\partial_{\lambda^*} + \lambda\partial_\lambda + \lambda^*\lambda(2n_B + 1)]\chi_j(\lambda, t). \tag{B6}$$

Now, the characteristic function of a thermal state with average occupation $n_{\mathcal{T}}(t)$ reads

$$\chi_{\mathcal{T}}(\lambda, t) = e^{-\lambda\lambda^*[n_{\mathcal{T}}(t)+1/2]}. \tag{B7}$$

Inserting this expression into Eq. (B4), we then find

$$\frac{d\chi_{\mathcal{T}}(\lambda, t)}{dt} = -\lambda\lambda^*\dot{n}_{\mathcal{T}}(t)\chi_{\mathcal{T}}(\lambda, t)$$
$$= \lambda\lambda^*\gamma(\omega_j(t))[n_{\mathcal{T}}(t) - n_B]\chi_{\mathcal{T}}(\lambda, t), \tag{B8}$$

which directly leads us to Eq. (4).

APPENDIX C: DERIVATION OF EQUATION (6)

We start by writing the Hamiltonian in Eq. (1) as

$$\hat{H}(t) = \hbar \vec{a}^{\dagger} \cdot \begin{pmatrix} \omega_s & \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^{\dagger} & \boldsymbol{\Omega}(t) \end{pmatrix} \cdot \vec{a}$$

$$= \hbar \vec{a}^{\dagger} \cdot \boldsymbol{H}(t) \cdot \vec{a}$$

$$= \hbar \sum_{m,n} H_{mn}(t)\hat{a}_m^{\dagger}\hat{a}_n, \qquad (C1)$$

where $\vec{a} = (\hat{a}_s, \hat{a}_1, \dots, \hat{a}_N)^{\mathrm{T}}$ contains the annihilation operators and we have defined the matrix $\boldsymbol{H}(t)$ reading

$$\boldsymbol{H}(t) = \begin{pmatrix} \omega_s & \alpha & \cdots & \alpha \\ \alpha & \omega_1 & \cdots & 0 \\ \alpha & 0 & \ddots & 0 \\ \alpha & 0 & \cdots & \omega_N \end{pmatrix}. \qquad (C2)$$

The equations of motion for the correlators, $\langle \hat{a}_k^{\dagger}\hat{a}_l \rangle = \mathrm{tr}\{\hat{a}_k^{\dagger}\hat{a}_l\hat{\rho}\}$, follow from Eq. (2) and read

$$\frac{d}{dt}\langle \hat{a}_k^{\dagger}\hat{a}_l \rangle = -\frac{i}{\hbar}\mathrm{tr}\{\hat{a}_k^{\dagger}\hat{a}_l[\hat{H},\hat{\rho}]\} + \gamma \sum_{j=1}^{N} \mathrm{tr}\{\hat{a}_k^{\dagger}\hat{a}_l\mathcal{D}_j\hat{\rho}\}$$

$$= -i\sum_{m,n} H_{mn}(\mathrm{tr}\{\hat{a}_k^{\dagger}\hat{a}_l\hat{a}_m^{\dagger}\hat{a}_n\hat{\rho}\} - \mathrm{tr}\{\hat{a}_m^{\dagger}\hat{a}_n\hat{a}_k^{\dagger}\hat{a}_l\hat{\rho}\})$$

$$+ \gamma \sum_{j=1}^{N}[1 + n_B(\omega_j)]\mathrm{tr}\left\{\hat{a}_k^{\dagger}\hat{a}_l\left(\hat{a}_j\hat{\rho}\hat{a}_j^{\dagger} - \frac{1}{2}\{\hat{a}_j^{\dagger}\hat{a}_j,\hat{\rho}\}\right)\right\} + \gamma \sum_{j=1}^{N} n_B(\omega_j)\mathrm{tr}\left\{\hat{a}_k^{\dagger}\hat{a}_l\left(\hat{a}_j^{\dagger}\hat{\rho}\hat{a}_j - \frac{1}{2}\{\hat{a}_j\hat{a}_j^{\dagger},\hat{\rho}\}\right)\right\}$$

$$= \sum_m[(iH_{km})\langle \hat{a}_m^{\dagger}\hat{a}_l \rangle + \langle \hat{a}_k^{\dagger}\hat{a}_m \rangle(-iH_{lm})] + \gamma[n_B(\omega_l)\delta_{kl} - \langle \hat{a}_k^{\dagger}\hat{a}_l \rangle]$$

$$= \sum_m(W_{km}\langle \hat{a}_m^{\dagger}\hat{a}_l \rangle + \langle \hat{a}_k^{\dagger}\hat{a}_m \rangle W_{ml}^*) + F_{kl}, \qquad (C3)$$

which is Eq. (6). Here, we have introduced the matrix $\boldsymbol{\Gamma} = \mathrm{diag}(0, \gamma/2, \gamma/2, \dots, \gamma/2)$ and defined

$$\boldsymbol{W} = i\boldsymbol{H} - \boldsymbol{\Gamma} \qquad (C4)$$

together with $\boldsymbol{F} = \mathrm{diag}(0, \gamma n_B(\omega_1), \dots, \gamma n_B(\omega_N))$.

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