Banerjee, Agnid; Garain, Prashanta; Kinnunen, Juha

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Lower semicontinuity and pointwise behavior of supersolutions for some doubly nonlinear nonlocal parabolic p-Laplace equations

Agnid Banerjee, Prashanta Garain and Juha Kinnunen

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Abstract

We discuss pointwise behavior of weak supersolutions for a class of doubly nonlinear parabolic fractional p-Laplace equations which includes the fractional parabolic p-Laplace equation and the fractional porous medium equation. More precisely, we show that weak supersolutions have lower semicontinuous representative. We also prove that the semicontinuous representative at an instant of time is determined by the values at previous times. This gives a pointwise interpretation for a weak supersolution at every point. The corresponding results hold true also for weak subsolutions. Our results extend of some recent results in the local parabolic case, and in the nonlocal elliptic case, to the nonlocal parabolic case. We prove the required energy estimates and measure theoretic De Giorgi type lemmas in the fractional setting.

Key words: Doubly nonlinear parabolic equation, fractional p-Laplace equation, porous medium equation, energy estimates, De Giorgi's method.

2010 Mathematics Subject Classification: 35K59, 35K65, 35B45, 35B65, 35R11.

1 Introduction

This article discusses pointwise behavior of weak supersolutions $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ to the doubly nonlinear parabolic nonlocal p-Laplace equation

$$\partial_t(|u|^{q-1}u) + \mathcal{L}u = 0 \text{ in } \Omega \times (0,T), \quad 1 < p < \infty, \quad q > 0, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $T > 0$ and the operator $\mathcal{L}$ is defined by

$$\mathcal{L}u(x,t) = \text{P.V.} \int_{\mathbb{R}^n} |u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))K(x,y,t)\,dy,$$

where P.V. stands for the principal value. We assume that the kernel K is a measurable symmetric kernel with respect to x and y and satisfies

$$\frac{\Lambda^{-1}}{|x-y|^{n+ps}} \le K(x,y,t) \le \frac{\Lambda}{|x-y|^{n+ps}}, \tag{1.2}$$

for almost every $x, y \in \mathbb{R}^n$ uniformly in $t \in (0, T)$ for some $\Lambda \geq 1$ and $s \in (0, 1)$. If $K(x, y, t) = |x - y|^{-(n+ps)}$, then $\mathcal{L}$ reduces, up to a multiplicative constant, to the fractional p-Laplace operator $(-\Delta)_p^s$ and (1.1) becomes

$$\partial_t(|u|^{q-1}u) + (-\Delta)_p^s u = 0. \tag{1.3}$$

For $p = 2$ and $q > 0$, we have an equation of porous medium type.

Our main result, Theorem 2.12, shows that a weak supersolution to (1.1) is lower semicontinuous after a possible redefinition on a set of measure zero. In other words, a weak supersolution to (1.1) has a lower semicontinuous representative. Furthermore, we prove that the lower semicontinuous representative can be recovered from the previous times, see Theorem 2.14. This gives a pointwise interpretation for a weak supersolution at every point. Our results applies to weak supersolutions and weak subsolutions which may change sign. In the nonlocal elliptic case, i.e.

$$\mathcal{L}u = 0 \text{ in } \Omega, \tag{1.4}$$

lower semicontinuity of weak supersolutions has been established by Korvenpää, Kuusi and Palatucci [15]. As far as we are aware, our work contains the first results in the nonlocal parabolic case. In the local case $s = 1$, lower semicontinuity of weak supersolutions to nonlinear parabolic problems has been studied by Kinnunen and Lindqvist [13, 14], Kuusi [16], Liao [18] and Ziemer [23].

We apply a general result of Liao in [18, Theorem 2.1], which asserts that every function that satisfies a measure theoretic property in Definition 2.10 has a lower semicontinuous representative. In [18] this result was applied for a large class of elliptic and parabolic partial differential equations in the local case $s = 1$, but we show that this technique can also be applied in the nonlocal case $s \in (0, 1)$. Our argument is based on a new energy estimate, Lemma 3.2, along with De Giorgi type lemmas, see Lemma 4.3 and Lemma 4.4. In particular, the approach is independent of Harnack type estimates. Lower semicontinuity results for weak supersolutions are needed, for example, in extending Perron's method in Korvenpää, Kuusi and Palatucci [15] to the parabolic nonlocal case, but this will be discussed elsewhere.

As far as we know, local Hölder continuity and Harnack's inequality for weak solutions to (1.1) are open questions. These questions have been studied extensively in the elliptic nonlocal case. For $p = 2$, Kassmann in [12] proved a Harnack inequality for equations of the type (1.4) with lower order perturbations. This was extended by Di Castro, Kuusi and Palatucci [8] to the case $1 < p < \infty$. See also Brasco, Lindgren and Schikorra [5] and Di Castro, Kuusi and Palatucci [7] for Hölder continuity results of (1.4).

When $q = 1$, the doubly nonlinear equation in (1.3) reduces to the fractional parabolic p-Laplace equation

$$\partial_t u + (-\Delta)_p^s u = 0. \tag{1.5}$$

For such an equation, Strömqvist in [22] established local boundedness estimate for subsolutions and Brasco, Lindgren and Strömqvist [6] proved Hölder continuity of weak solutions to

(1.5) for $p \geq 2$. In the case when $p = 2$, i.e. for equations modeled on $\partial_t u + (-\Delta)^s u = 0$, Harnack's inequality has been established by Felsinger and Kassmann in [10], see also Strömqvist [21]. Finally we mention that qualitative properties for fractional porous medium equations of the type,

$$\partial_t u + (-\Delta)^s (u^m) = 0, \quad m > 0,$$

has been studied by Bonforte, Sire and Vázquez [3], Bonforte and Vázquez [4], de Pablo, Quirós, Rodríguez and Vázquez [20]. When $q = p - 1$, equation (1.3) becomes the doubly nonlinear equation

$$\partial_t (|u|^{p-2} u) + (-\Delta)_p^s u = 0.$$

This equation is homogeneous in the sense that the class of solutions is closed under multiplication by constants. For such an equation, local boundedness estimate for positive subsolutions and a reverse Hölder inequality for positive supersolutions has been proven by Banerjee, Garain and Kinnunen [1].

This article is organized as follows. In Section 2, we introduce basic notation, gather some preliminary results that are relevant to our work and then state the main results. In Section 3, we prove an energy estimate which is needed to derive the measure theoretic De Giorgi type lemmas in Section 4 using which we prove our main results.

2 Preliminaries and main results

We begin with some known results for fractional Sobolev spaces, see [19] for more details.

Definition 2.1 *Let $1 < p < \infty$ and $0 < s < 1$ and assume that $\Omega \subset \mathbb{R}^n$ is an open and connected subset of $\mathbb{R}^n$. The fractional Sobolev space $W^{s,p}(\Omega)$ is defined by*

$$W^{s,p}(\Omega) = \Big\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \Big\}$$

and it is endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big(\int_\Omega |u(x)|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} \, dx \, dy \Big)^{\frac{1}{p}}.$$

The fractional Sobolev space with zero boundary values is defined by

$$W_0^{s,p}(\Omega) = \big\{ u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \big\}.$$

Both $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ are reflexive Banach spaces, see [19]. For any $v, k \in \mathbb{R}$, we denote the positive and negative parts of $(v - k)$ by

$$(v-k)_+ = \max\{(v-k), 0\} \quad \text{and} \quad (v-k)_- = \max\{-(v-k), 0\}.$$

We note that $(v-k)_- = (k-v)_+$. For any $a, b \in \mathbb{R}$, we have $|a_+ - b_+| \leq |a - b|$ which implies $u_+ \in W^{s,p}(\Omega)$ when $u \in W^{s,p}(\Omega)$. Analogously, we have $u_- \in W^{s,p}(\Omega)$.

The parabolic Sobolev space $L^p(0,T;W^{s,p}(\Omega))$, $T>0$, consists of measurable functions u on $\Omega\times(0,T)$ such that

$$||u||_{L^p(0,T;W^{s,p}(\Omega))}=\Big(\int_0^T ||u(\cdot,t)||^p_{W^{s,p}(\Omega)}\,dt\Big)^{\frac{1}{p}}<\infty.$$

The local space $L^p_{\text{loc}}(0,T;W^{s,p}_{\text{loc}}(\Omega))$ is defined by requiring the conditions above for every $\Omega'\times[t_1,t_2]\Subset\Omega\times(0,T)$.

Next we state a Sobolev embedding theorem, see [19].

Theorem 2.2 *Let $1<p<\infty$ and $0<s<1$ with $ps<n$ and $\kappa^*=\frac{n}{n-ps}$. There exists a constant $C=C(n,p,s)$ such that*

$$\|u\|^p_{L^{\kappa^*p}(\mathbb{R}^n)}\le C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy$$

for every $u\in W^{s,p}(\mathbb{R}^n)$. If Ω is a bounded $W^{s,p}$-extension domain, there exists a constant $C=C(n,s,p,\Omega)$ such that, for any $\kappa\in[1,\kappa^*]$, we have

$$\|u\|_{L^{\kappa p}(\Omega)}\le C||u||_{W^{s,p}(\Omega)}$$

for every $u\in W^{s,p}(\Omega)$.
If $ps=n$, then the above inequalities hold for any $\kappa\in[1,\infty)$. For $ps>n$, the second inequality holds for any $\kappa\in[1,\infty]$.

Let $B_r(x_0)=\{x\in\mathbb{R}^n:|x-x_0|<r\}$ denote the ball in $\mathbb{R}^n$ of radius $r>0$ and center at $x_0\in\mathbb{R}^n$. The barred integral sign denotes the corresponding integral average. As an application of Theorem 2.2 we obtain the following Sobolev type inequality, see [22, Lemma 2.1] for a proof.

Lemma 2.3 *Let $1<p<\infty$ and $0<s<1$. Assume that $u\in W^{s,p}(B_r)$, where $B_r=B_r(x_0)$. Let $\kappa^*=\frac{n}{n-ps}$, if $ps<n$. There exists a constant $C=C(n,p,s)$ such that for every $\kappa\in[1,\kappa^*]$, we have*

$$\Big(\fint_{B_r}|u(x)|^{\kappa p}\,dx\Big)^{\frac{1}{\kappa}}\le Cr^{ps-n}\int_{B_r}\int_{B_r}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy+C\fint_{B_r}|u(x)|^p\,dx. \tag{2.1}$$

If $ps\ge n$, let $N>0$ be such that $n\le ps<N$. Then the inequality (2.1) *holds for every $\kappa\in[1,\kappa^*]$ where $\kappa^*=\frac{N}{N-ps}$.*

The next inequality is a straightforward consequence of Hölder's inequality.

Lemma 2.4 *Let $1<p<\infty$ and $0<s<1$. Assume that $u\in L^p(t_1,t_2;W^{s,p}(B_r))\cap L^\infty(t_1,t_2;L^m(B_r))$ where $m\ge1$. Let $\kappa^*=\frac{n}{n-ps}$, if $ps<n$. Then for $l=p(1+\frac{ms}{n})$, we have*

$$\int_{t_1}^{t_2}\int_{B_r}|u|^l\,dxdt\le\int_{t_1}^{t_2}\Big(\int_{B_r}|u|^{p\kappa^*}\,dx\Big)^{\frac{1}{\kappa^*}}\,dt\Big(\operatorname*{ess\,sup}_{t_1<t<t_2}\int_{B_r}|u|^m\,dx\Big)^{\frac{ps}{n}}. \tag{2.2}$$

If $ps\ge n$, let $N>0$ be such that $n\le ps<N$. Then the inequality (2.2) *holds with n, l and κ^* replaced by N, $l=p(1+\frac{ms}{N})$ and $\kappa^*=\frac{N}{N-ps}$ respectively.*

We fix notation that will be used throughout the rest of the paper. Let $\Omega' \subset \Omega$ and $0 \leq t_1 < t_2 \leq T$. We define the parabolic boundary of a space-time cylinder $\Omega'_{t_1,t_2} := \Omega' \times (t_1, t_2)$ by

$$\partial_p \Omega'_{t_1,t_2} := (\overline{\Omega'} \times \{t_1\}) \cup (\partial\Omega' \times [t_1, t_2]).$$

Let $r, \theta > 0$ and $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$. Then we denote the backward and forward in time cylinders by

$$\begin{aligned} \mathcal{Q}^-(x_0, t_0; r, \theta) &:= B_r(x_0) \times (t_0 - \theta r^{sp}, t_0], \\ \mathcal{Q}^+(x_0, t_0; r, \theta) &:= B_r(x_0) \times [t_0, t_0 + \theta r^{sp}), \end{aligned} \tag{2.3}$$

respectively. The centered cylinder is denoted by

$$\mathcal{Q}_r(x_0, t_0; \theta) := \mathcal{Q}^-(x_0, t_0; r, \theta) \cup \mathcal{Q}^+(x_0, t_0; r, \theta). \tag{2.4}$$

When $(x_0, t_0) = (0, 0)$, we simply denote $\mathcal{Q}^-(x_0, t_0; r, \theta)$ and $\mathcal{Q}^+(x_0, t_0; r, \theta)$ by $\mathcal{Q}_r^-(\theta)$ and $\mathcal{Q}_r^+(\theta)$, respectively. Also when $\theta = 1$, we denote $\mathcal{Q}_r(x_0, t_0; \theta)$ by $\mathcal{Q}_r(x_0, t_0)$.

For short, we denote

$$\mathcal{A}(u(x, y, t)) := |u(x, t) - u(y, t)|^{p-2}(u(x, t) - u(y, t)) \quad \text{and} \quad d\mu := K(x, y, t)\, dx\, dy.$$

Next we define the notion of weak solution of the doubly nonlinear parabolic nonlocal equation in (1.1).

Definition 2.5 *A function $u \in L^\infty(0, T; L^\infty(\mathbb{R}^n))$ is a weak supersolution (subsolution) to (1.1) if $u \in L^{q+1}_{\mathrm{loc}}(0, T; L^{q+1}_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(0, T; W^{s,p}_{\mathrm{loc}}(\Omega))$ and for every $\Omega' \times [t_1, t_2] \Subset \Omega \times (0, T)$, and nonnegative test function $\phi \in W^{1,q+1}_{\mathrm{loc}}(0, T; L^{q+1}(\Omega')) \cap L^p_{\mathrm{loc}}(0, T; W^{s,p}_0(\Omega'))$, we have*

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{A}(u(x, y, t))(\phi(x, t) - \phi(y, t))\, d\mu\, dt - \int_{t_1}^{t_2} \int_{\Omega'} |u(x, t)|^{q-1} u(x, t) \partial_t \phi(x, t)\, dx\, dt \\ + \int_{\Omega'} |u(x, t_2)|^{q-1} u(x, t_2) \phi(x, t_2)\, dx - \int_{\Omega'} |u(x, t_1)|^{q-1} u(x, t_1) \phi(x, t_1)\, dx \geq 0\, (\leq 0). \end{aligned} \tag{2.5}$$

We say that u is a weak solution if the integral in (2.5) is zero for every test function without a sign restriction.

Remark 2.6 *The boundary terms in (2.5) are understood in the sense that*

$$\int_{\Omega'} |u(x, t_1)|^{q-1} u(x, t_1) \phi(x, t_1)\, dx = \lim_{h \to 0} \fint_{t_1}^{t_1+h} \int_{\Omega'} |u(x, t)|^{q-1} u(x, t) \phi(x, t)\, dx\, dt,$$

and

$$\int_{\Omega'} |u(x, t_2)|^{q-1} u(x, t_2) \phi(x, t_2)\, dx = \lim_{h \to 0} \fint_{t_2-h}^{t_2} \int_{\Omega'} |u(x, t)|^{q-1} u(x, t) \phi(x, t)\, dx\, dt.$$

Remark 2.7 *From Definition 2.5, it follows that u is a weak supersolution of* (1.1) *if and only if $-u$ is a weak subsolution of* (1.1). *Moroever, u is a weak solution if it is both a weak supersolution and a weak subsolution.*

Remark 2.8 *The assumption $u \in L^\infty(0,T;L^\infty(\mathbb{R}^n))$ ensures that the first term in the left hand side of* (2.5) *is finite.*

Remark 2.9 *In the next section, we prove energy estimates for a weak supersolution u, where we use test functions depending on u itself. The time derivative u_t can be justified by using a mollification*

$$f_h(x,t) = \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} f(x,s)\,ds. \tag{2.6}$$

in time (see [1, 2, 14]).

Let u be a measurable function which is locally essentially bounded below in $\Omega \times (0,T)$. We define the lower semicontinuous regularization u_* of u as

$$u_*(x,t) := \operatorname*{ess\,lim\,inf}_{(y,\hat{t})\to(x,t)} u(y,\hat{t}) = \lim_{r\to 0} \operatorname*{ess\,inf}_{\mathcal{Q}_r(x,t;\theta)} u$$

for every $(x,t) \in \Omega \times (0,T)$. Analogously, for a locally essentially bounded above measurable function u in $\Omega \times (0,T)$, we define an upper semicontinuous regularization u^* of u by

$$u^*(x,t) := \operatorname*{ess\,lim\,sup}_{(y,\hat{t})\to(x,t)} u(y,\hat{t}) = \lim_{r\to 0} \operatorname*{ess\,sup}_{\mathcal{Q}_r(x,t;\theta)} u$$

for every $(x,t) \in \Omega \times (0,T)$. It is easy to see that u_* is lower semicontinuous and u^* is upper semicontinuous in $\Omega \times (0,T)$.

The following measure theoretic property from [18] will be useful for us.

Definition 2.10 *Let u be a measurable function which is locally bounded from below in $\Omega \times (0,T)$ and let*

$$\mu_- \leq \operatorname*{ess\,inf}_{\mathcal{Q}_r(x_0,t_0;\theta)} u.$$

Moreover, let $a, c \in (0,1)$ and $M > 0$. We say that u satisfies the property $(\mathcal{D})$ if there exists a constant $\tau \in (0,1)$, which depends only on a, M, μ_- and other data, but independent of r, such that

$$|\{u \leq \mu_- + M\} \cap \mathcal{Q}_r(x_0,t_0;\theta)| \leq \tau |\mathcal{Q}_r(x_0,t_0;\theta)|$$

implies

$$u \geq \mu_- + aM \ \text{ a.e. in } \mathcal{Q}_{cr}(x_0,t_0;\theta).$$

Let $u \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$ and define

$$\mathcal{F} := \Big\{(x,t) \in \Omega \times (0,T) : |u(x,t)| < \infty,\ \lim_{r\to 0} \fint_{\mathcal{Q}_r(x,t;\theta)} |u(x,t) - u(y,\hat{t})|\,dy\,d\hat{t} = 0\Big\}.$$

From the Lebesgue differentiation theorem we have $|\mathcal{F}| = |\Omega \times (0,T)|$. A result of Liao in [18, Theorem 2.1] shows that any such function with the property $(\mathcal{D})$ has a lower semicontinuous representative.

Theorem 2.11 *Let u be a measurable function in $\Omega \times (0,T)$ which is locally essentially bounded below in $\Omega \times (0,T)$ and satisfies the property $(\mathcal{D})$. Then $u(x,t) = u_*(x,t)$ for every $(x,t) \in \mathcal{F}$. In particular, u_* is a lower semicontinuous representative of u in $\Omega \times (0,T)$.*

Next we state our main results. Our first result is a lower semicontinuity result for weak supersolutions. This follows from Lemma 4.3 below, which asserts that weak supersolutions satisfy the property $(\mathcal{D})$, and Theorem 2.11.

Theorem 2.12 *Let $1 < p < \infty$, $q > 0$ and u be a weak supersolution of (1.1) in $\Omega \times (0,T)$. Then $u_*(x,t) = u(x,t)$ at every Lebesgue point $(x,t) \in \Omega \times (0,T)$. In particular, u_* is a lower semicontinuous representative of u in $\Omega \times (0,T)$.*

As an immediate corollary we have the corresponding result for weak subsolutions.

Corollary 2.13 *Let $1 < p < \infty$, $q > 0$ and u be a weak subsolution of (1.1) in $\Omega \times (0,T)$. Then $u^*(x,t) = u(x,t)$ at every Lebesgue point $(x,t) \in \Omega \times (0,T)$. In particular, u^* is an upper semicontinuous representative of u in $\Omega \times (0,T)$.*

The second result asserts that the lower semicontinuous representative, given by Theorem 2.12, is determined by previous times. The proof is based on Lemma 4.4 below and the proof of [18, Theorem 3.1].

Theorem 2.14 *Let $1 < p < \infty$, $q > 0$ and u be a weak supersolution of (1.1) in $\Omega \times (0,T)$ and u_* is the lower semicontinuous representative of u given by Theorem 2.12. Then there exists $\theta > 0$ such that for every $(x,t) \in \Omega \times (0,T)$, we have*

$$u_*(x,t) = \inf_{\theta>0} \lim_{r\to 0} \operatorname*{ess\,inf}_{\mathcal{Q}'_r(x,t,\theta)} u,$$

where $\mathcal{Q}'_r(x,t,\theta) = B_r(x) \times (t - 2\theta r^{ps}, t - \theta r^{ps})$. As a consequence,

$$u_*(x,t) = \operatorname*{ess\,lim\,inf}_{(y,\hat{t})\to(x,t),\,\hat{t}<t} u(y,\hat{t})$$

at every point $(x,t) \in \Omega \times (0,T)$.

3 The energy estimate

In order to be able to consider sign changing functions, for $a \in \mathbb{R}$ and $\beta > 0$, we define

$$a^\beta = \begin{cases} |a|^{\beta-1}a, & a \neq 0, \\ 0, & a = 0. \end{cases} \tag{3.1}$$

In addition, for $a, k \in \mathbb{R}$ and $q > 0$, we define a nonnegative auxiliary function ξ by

$$\xi((a-k)_-) = q \int_a^k |\eta|^{q-1} (\eta - k)_- \, d\eta. \tag{3.2}$$

For more applications of this kind of function in the doubly nonlinear context, we refer to [2, 11, 17, 18].

The following elementary inequality will be useful for us, see [7, Lemma 3.1].

Lemma 3.1 *Let $p \geq 1$ and $\gamma \in (0,1]$. For every $a, b \in \mathbb{R}^n$, we have*

$$|a|^p \leq |b|^p + C(p)\gamma |b|^p + \big(1 + C(p)\gamma\big)\gamma^{1-p} |a-b|^p,$$

where $C(p) = (p-1)\Gamma(\max\{1, p-2\})$ and Γ denotes the gamma function.

To prove Theorem 2.12 and Theorem 2.14, we apply the following Caccioppoli type estimate for weak supersolutions.

Lemma 3.2 *Let $1 < p < \infty$, q, $l, > 0$, $k \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ be such that $B_r \times (t_0 - l, t_0) := B_r(x_0) \times (t_0 - l, t_0) \Subset \Omega \times (0, T)$. Assume that u is a weak supersolution of (1.1) in $\Omega \times (0,T)$. Then there exists a constant $C = C(p, \Lambda)$ such that*

$$\begin{aligned}
&\int_{t_0-l}^{t_0} \int_{B_r} \int_{B_r} |(u-k)_-(x,t)\psi(x,t) - (u-k)_-(y,t)\psi(y,t)|^p \, d\mu \, dt \\
&\qquad + \operatorname*{ess\,sup}_{t_0 - l < t < t_0} \int_{B_r} \psi(x,t)^p \xi((u-k)_-(x,t)) \, dx \\
&\leq C \Bigg(\int_{t_0-l}^{t_0} \int_{B_r} \int_{B_r} \max\{(u-k)_-(x,t), (u-k)_-(y,t)\}^p |\psi(x,t) - \psi(y,t)|^p \, d\mu \, dt \\
&\qquad + \operatorname*{ess\,sup}_{(x,t) \in \operatorname{supp}\psi,\, t_0 - l < t < t_0} \int_{\mathbb{R}^n \setminus B_r} \frac{(u-k)_-(y,t)^{p-1}}{|x-y|^{n+ps}} \, dy \int_{t_0-l}^{t_0} \int_{B_r} (u-k)_-(x,t)\psi(x,t)^p \, dx \, dt \\
&\qquad + \int_{t_0-l}^{t_0} \int_{B_r} \xi((u-k)_-(x,t)) \partial_t \psi(x,t)^p \, dx \, dt + \int_{B_r \times \{t_0 - l\}} \psi(x,t)^p \xi((u-k)_-(x,t)) \, dx \Bigg),
\end{aligned}$$

for every nonnegative, piecewise smooth cutoff function ψ vanishing on $\partial B_r \times (t_0 - l, t_0)$. Here ξ is the auxiliary function defined in (3.2).

Proof. We denote by $t_1 = t_0 - l$, $t_2 = t_0$ and $w(x,t) = (u-k)_-(x,t) = (k-u)_+(x,t)$. Following [2], for fixed $t_1 < l_1 < l_2 < t_2$ and $\epsilon > 0$ small enough, we define a Lipschitz continuous cutoff function $\zeta_\epsilon : [t_1, t_2] \to [0,1]$ by

$$\zeta_\epsilon(t) := \begin{cases} 0, & t_1 \leq t \leq l_1 - \epsilon, \\ 1 + \frac{t - l_1}{\epsilon}, & l_1 - \epsilon < t \leq l_1, \\ 1, & l_1 < t \leq l_2, \\ 1 - \frac{t - l_2}{\epsilon}, & l_2 < t \leq l_2 + \epsilon, \\ 0, & l_2 + \epsilon < t \leq t_2. \end{cases}$$

Choose

$$\phi(x,t) = w(x,t)\psi(x,t)^p\zeta_\epsilon(t)$$

as a test function in (2.5). Keeping in mind (3.1), we denote

$$v_h^q(x,t) = (u^q)_h(x,t) \quad \text{and} \quad \mathcal{V}(u(x,y,t)) = \mathcal{A}(u(x,y,t))K(x,y,t),$$

where $(\cdot)_h$ is the mollification as defined in (2.6). Following [1, 2, 14], we conclude that

$$\lim_{\epsilon\to 0}\lim_{h\to 0}(I_{h,\epsilon} + J_{h,\epsilon}) \geq 0, \tag{3.3}$$

where

$$I_{h,\epsilon} = \int_{t_1}^{t_2}\int_{B_r} \partial_t v_h^q(x,t)\phi(x,t)\,dx\,dt = \int_{t_1}^{t_2}\int_{B_r} \psi(x,t)^p\zeta_\epsilon(t)\partial_t v_h^q(x,t)w(x,t)\,dx\,dt,$$

and

$$\begin{aligned} J_{h,\epsilon} &= \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\mathcal{V}(u(x,y,t))\big)_h(\phi(x,t)-\phi(y,t))\,dx\,dy\,dt \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\mathcal{V}(u(x,y,t))\big)_h\big(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p\big)\zeta_\epsilon(t)\,dx\,dy\,dt. \end{aligned}$$

Estimate of $I_{h,\epsilon}$: Proceeding similarly as in the proof of [2, Proposition 3.1], we may rewrite

$$\begin{aligned} I_{h,\epsilon} = \int_{t_1}^{t_2}\int_{B_r} &\psi(x,t)^p\zeta_\epsilon(t)\partial_t v_h^q(x,t)\big(w(x,t)-(k-v_h)_+(x,t)\big)\,dx\,dt \\ &+ \int_{t_1}^{t_2}\int_{B_r} \psi(x,t)^p\zeta_\epsilon(t)\partial_t v_h^q(x,t)(k-v_h)_+(x,t)\,dx\,dt. \end{aligned} \tag{3.4}$$

By the properties of mollifiers, we have

$$\partial_t v_h^q = \frac{u^q - v_h^q}{h}.$$

Then by using the fact that the mapping $x \to (k - x^{\frac{1}{q}})_+$ is monotone decreasing and also by noting that $w = (k-u)_+$, we conclude that the first integral in (3.4) is nonpositive and thus

$$\begin{aligned} I_{h,\epsilon} &\leq \int_{t_1}^{t_2}\int_{B_r} \psi(x,t)^p\zeta_\epsilon(t)\partial_t v_h^q(x,t)(k-v_h)_+(x,t)\,dx\,dt \\ &= -\int_{t_1}^{t_2}\int_{B_r} \psi(x,t)^p\zeta_\epsilon(t)\partial_t\xi\big((k-v_h)_+\big)\,dx\,dt \\ &= \int_{t_1}^{t_2}\int_{B_r} \big(\psi(x,t)^p\zeta_\epsilon'(t) + \partial_t\psi(x,t)^p\zeta_\epsilon(t)\big)\xi\big((k-v_h)_+(x,t)\big)\,dx\,dt, \end{aligned} \tag{3.5}$$

where we have used the fact that

$$\partial_t\xi((k-v_h)_+) = -\partial_t v_h^q(k-v_h)_+.$$

By passing to the limit as $h \to 0$ in (3.5), we obtain

$$\lim_{h\to 0} I_{h,\epsilon} \leq \int_{t_1}^{t_2}\int_{B_r} \big(\psi(x,t)^p\zeta_\epsilon'(t) + \partial_t\psi(x,t)^p\zeta_\epsilon(t)\big)\xi(w)\,dx\,dt. \tag{3.6}$$

Then by letting $\epsilon \to 0$ in (3.6), we have

$$\begin{aligned}\lim_{\epsilon\to 0}\lim_{h\to 0} I_{h,\epsilon} \leq \int_{B_r}\psi(x,l_1)^p\xi(w(x,l_1))\,dx - \int_{B_r}\psi(x,l_2)^p\xi(w(x,l_2))\,dx \\ + \int_{l_1}^{l_2}\int_{B_r}\partial_t\psi(x,t)^p\xi(w(x,t))\,dx\,dt.\end{aligned} \tag{3.7}$$

Estimate of $J_{h,\epsilon}$: Following the proof of [1, Lemma 3.1], we obtain

$$\begin{aligned}\lim_{h\to 0} J_{h,\epsilon} &= J_\epsilon \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathcal{V}(u(x,y,t))\big(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p\big)\zeta_\epsilon(t)\,dx\,dy\,dt \\ &= J_\epsilon^1 + J_\epsilon^2,\end{aligned} \tag{3.8}$$

where

$$J_\epsilon^1 = \int_{t_1}^{t_2}\int_{B_r}\int_{B_r}\mathcal{A}(u(x,y,t))\big(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p\big)\zeta_\epsilon(t)\,d\mu\,dt,$$

and

$$J_\epsilon^2 = 2\int_{t_1}^{t_2}\int_{\mathbb{R}^n\setminus B_r}\int_{B_r}\mathcal{A}(u(x,y,t))w(x,t)\psi(x,t)^p\zeta_\epsilon(t)\,d\mu\,dt.$$

Estimate of J_ϵ^1: To estimate the integral J_ϵ^1, we use some ideas from the proof of [7, Theorem 1.4]. By symmetry we may assume $u(x,t) < u(y,t)$. In this case, for every fixed t, we observe that

$$\begin{aligned}&|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))\big(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p\big) \\ &= -(u(y,t)-u(x,t))^{p-1}\big(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p\big) \\ &= \begin{cases} -(w(x,t)-w(y,t))^{p-1}\big(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p\big), & \text{if } u(x,t)<k,\ u(y,t)<k, \\ -(u(y,t)-u(x,t))^{p-1}w(x,t)\psi(x,t)^p, & \text{if } u(x,t)\leq k,\ u(y,t)>k, \\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

where we used the fact that $u(x,t) < u(y,t)$ implies $w(x,t) > w(y,t)$. Therefore in all cases we have

$$\begin{aligned}&|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))\big(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p\big) \\ &\quad\leq -(w(x,t)-w(y,t))^{p-1}\big(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p\big).\end{aligned}$$

This implies

$$J_\epsilon^1 \leq -\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}(w(x,t)-w(y,t))^{p-1}(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p)\zeta_\epsilon(t)\,d\mu\,dt.$$

Let us first consider the case when $\psi(x,t) \le \psi(y,t)$. Choosing $a = \psi(y,t)$ and $b = \psi(x,t)$ in Lemma 3.1 we obtain

$$\psi(x,t)^p \ge (1 - C(p)\gamma)\psi(y,t)^p - (1 + C(p)\gamma)\gamma^{1-p}|\psi(x,t) - \psi(y,t)|^p \tag{3.9}$$

for any $\gamma \in (0,1]$, where $C(p) = (p-1)\Gamma(\max\{1, p-2\})$. By letting

$$\gamma = \frac{1}{\max\{1, 2C(p)\}} \frac{w(x,t) - w(y,t)}{w(x,t)} \in (0,1],$$

we deduce from (3.9) that there exists a positive constant $C = C(p)$ such that

$$\begin{aligned}
&(w(x,t) - w(y,t))^{p-1} w(x,t)\psi(x,t)^p \\
&\quad \ge (w(x,t) - w(y,t))^{p-1} w(x,t) \max\{\psi(x,t), \psi(y,t)\}^p \\
&\qquad - \frac{1}{2}(w(x,t) - w(y,t))^p \max\{\psi(x,t), \psi(y,t)\}^p \\
&\qquad - C \max\{w(x,t), w(y,t)\}^p |\psi(x,t) - \psi(y,t)|^p.
\end{aligned}$$

Here we used the fact that under the assumptions $\psi(x,t) \le \psi(y,t)$ and $w(x,t) > w(y,t)$ we have $\max\{\psi(x,t), \psi(y,t)\} = \psi(y,t)$ and $\max\{w(x,t), w(y,t)\} = w(x,t)$. In the other cases $w(x,t) > w(y,t)$, $\psi(x,t) \ge \psi(y,t)$ or $w(x,t) = w(y,t)$, the above estimate is clear. Therefore, when $w(x,t) \ge w(y,t)$, we have

$$\begin{aligned}
&(w(x,t) - w(y,t))^{p-1}(w(x,t)\psi(x,t)^p - w(y,t)\psi(y,t)^p) \\
&\quad \ge (w(x,t) - w(y,t))^{p-1}(w(x,t) \max\{\psi(x,t), \psi(y,t)\}^p - w(y,t)\psi(y,t)^p) \\
&\qquad - \frac{1}{2}(w(x,t) - w(y,t))^p \max\{\psi(x,t), \psi(y,t)\}^p \\
&\qquad - C \max\{w(x,t), w(y,t)\}^p |\psi(x,t) - \psi(y,t)|^p \\
&\quad \ge \frac{1}{2}(w(x,t) - w(y,t))^p \max\{\psi(x,t), \psi(y,t)\}^p \\
&\qquad - C \max\{w(x,t), w(y,t)\}^p |\psi(x,t) - \psi(y,t)|^p,
\end{aligned} \tag{3.10}$$

where $C = C(p)$. If $w(x,t) < w(y,t)$, we may interchange the roles of x and y above in order to obtain (3.10). We observe that

$$\begin{aligned}
|w(x,t)\psi(x,t) - w(y,t)\psi(y,t)|^p &\le 2^{p-1}|w(x,t) - w(y,t)|^p \max\{\psi(x,t), \psi(y,t)\}^p \\
&\quad + 2^{p-1} \max\{w(x,t), w(y,t)\}^p |\psi(x,t) - \psi(y,t)|^p.
\end{aligned} \tag{3.11}$$

Thus from (3.10) and (3.11), we deduce that

$$\begin{aligned}
J_\epsilon^1 &\le -c \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} |w(x,t)\psi(x,t) - w(y,t)\psi(y,t)|^p \zeta_\epsilon(t)\, d\mu\, dt \\
&\quad + C \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \max\{w(x,t), w(y,t)\}^p |\psi(x,t) - \psi(y,t)|^p \zeta_\epsilon(t)\, d\mu\, dt,
\end{aligned} \tag{3.12}$$

for some positive constants $c = c(p)$ and $C = C(p)$.

Estimate of J_ϵ^2: To estimate J_ϵ^2, let $x \in B_r$ and $y \in \mathbb{R}^n \setminus B_r$. If $u(x,t) \leq u(y,t)$, then $J_\epsilon^2 \leq 0$. It remains to consider the case when $u(x,t) > u(y,t)$. If $u(y,t) < u(x,t) < k$, then

$$\begin{aligned}|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))w(x,t) &= (u(x,t)-u(y,t))^{p-1}w(x,t)\\ &\leq w(y,t)^{p-1}w(x,t).\end{aligned}$$

This implies

$$\begin{aligned}J_\epsilon^2 &\leq \int_{t_1}^{t_2}\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} K(x,y,t)w(y,t)^{p-1}w(x,t)\psi(x,t)^p\zeta_\epsilon(t)\,dx\,dy\,dt\\ &\leq \Lambda \operatorname*{ess\,sup}_{(x,t)\in\operatorname{supp}\psi,\, t_1<t<t_2}\int_{\mathbb{R}^n\setminus B_r}\frac{w(y,t)^{p-1}}{|x-y|^{n+ps}}\,dy\int_{t_1}^{t_2}\int_{B_r} w(x,t)\psi(x,t)^p\zeta_\epsilon(t)\,dx\,dt.\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we conclude that there exist positive constants $c = c(p)$ and $C = C(p)$, such that

$$\begin{aligned}&\lim_{\epsilon\to 0}\lim_{h\to 0} J_{h,\epsilon} = \lim_{\epsilon\to 0} J_\epsilon = \lim_{\epsilon\to 0}(J_\epsilon^1 + J_\epsilon^2)\\ &\leq -c\int_{l_1}^{l_2}\int_{B_r}\int_{B_r}|w(x,t)\psi(x,t)-w(y,t)\psi(y,t)|^p\,d\mu\,dt\\ &\quad + C\int_{l_1}^{l_2}\int_{B_r}\int_{B_r}\max\{w(x,t),w(y,t)\}^p|\psi(x,t)-\psi(y,t)|^p\,d\mu\,dt\\ &\quad + \Lambda \operatorname*{ess\,sup}_{(x,t)\in\operatorname{supp}\psi,\, t_1<t<t_2}\int_{\mathbb{R}^n\setminus B_r}\frac{w(y,t)^{p-1}}{|x-y|^{n+ps}}\,dy\int_{l_1}^{l_2}\int_{B_r} w(x,t)\psi(x,t)^p\,dx\,dt.\end{aligned} \tag{3.14}$$

By inserting (3.7) and (3.14) into (3.3) and first by letting $l_1 \to t_1$ and then by letting $l_2 \to t_2$, we get

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}|w(x,t)\psi(x,t)-w(y,t)\psi(y,t)|^p\,d\mu\,dt\\ &\leq C(p,\Lambda)\Bigg(\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}\max\{w(x,t),w(y,t)\}^p|\psi(x,t)-\psi(y,t)|^p\,d\mu\,dt\\ &\qquad + \operatorname*{ess\,sup}_{(x,t)\in\operatorname{supp}\psi,\, t_1<t<t_2}\int_{\mathbb{R}^n\setminus B_r}\frac{w(y,t)^{p-1}}{|x-y|^{n+ps}}\,dy\int_{t_1}^{t_2}\int_{B_r} w(x,t)\psi(x,t)^p\,dx\,dt\\ &\qquad + \int_{t_1}^{t_2}\int_{B_r}\xi(w(x,t))\partial_t\psi(x,t)^p\,dx\,dt + \int_{B_r\times\{t_1\}}\psi(x,t)^p\xi(w(x,t))\,dx\Bigg).\end{aligned} \tag{3.15}$$

Again using (3.7) and (3.14) and first by letting $l_1 \to t_1$ and then by choosing $l_2 \in (t_1, t_2)$ such that

$$\int_{B_r}\xi(w(x,l_2))\psi(x,l_2)^p\,dx \geq \frac{1}{2}\operatorname*{ess\,sup}_{t_1<t<t_2}\int_{B_r}\xi(w(x,t))\psi(x,t)^p\,dx,$$

we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_1<t<t_2} \int_{B_r} \xi(w(x,t))\psi(x,t)^p\,dx \\
&\le C(p,\Lambda)\Bigg(\int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \max\{w(x,t),w(y,t)\}^p |\psi(x,t)-\psi(y,t)|^p\,d\mu\,dt \\
&\qquad + \operatorname*{ess\,sup}_{(x,t)\in\operatorname{supp}\psi,\, t_1<t<t_2} \int_{\mathbb{R}^n\setminus B_r} \frac{w(y,t)^{p-1}}{|x-y|^{n+ps}}\,dy \int_{t_1}^{t_2}\int_{B_r} w(x,t)\psi(x,t)^p\,dx\,dt \\
&\qquad + \int_{t_1}^{t_2}\int_{B_r} \xi(w(x,t))\partial_t\psi(x,t)^p\,dx\,dt + \int_{B_r\times\{t_1\}} \psi(x,t)^p\xi(w(x,t))\,dx \Bigg).
\end{aligned} \tag{3.16}$$

The required estimate follows from (3.15) and (3.16).

4 Proof of the main results

From [2, Lemma 2.2], we have the following bounds for the auxiliary function ξ defined in (3.2).

Lemma 4.1 *Let $q > 0$. Then there exists a constant $\lambda = \lambda(q) > 0$ such that*

$$\frac{1}{\lambda}(|a|+|k|)^{q-1}(a-k)_-^2 \le \xi((a-k)_-) \le \lambda(|a|+|k|)^{q-1}(a-k)_-^2$$

for every $a, k \in \mathbb{R}$.

For the following real analysis lemma, see [9, Lemma 4.1].

Lemma 4.2 *Let $(Y_j)_{j=0}^\infty$ be a sequence of positive real numbers such that for some constants $c_0 > 0$, $b > 1$ and $\beta > 0$ we have*

$$Y_0 \le c_0^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}} \quad \textit{and} \quad Y_{j+1} \le c_0 b^j Y_j^{1+\beta} \quad \textit{for every } j = 0,1,2,\dots.$$

Then $Y_j \to 0$ as $j \to \infty$.

We now prove a De Giorgi type lemma from which it follows that a weak supersolution satisfies the property $(\mathcal{D})$ in Definition 2.10.

Lemma 4.3 *Let $1 < p < \infty$, q, L, $\theta > 0$, $a \in (0,1)$ and assume that u is a weak supersolution of (1.1) in $\Omega \times (0,T)$. Let $\mathcal{Q}^-(x_0,t_0;r,\theta) \Subset \Omega\times(0,T)$ and assume that $\mu_- \le \operatorname{ess\,inf}_{\mathcal{Q}_r(x_0,t_0;\theta)} u$, $\lambda_- \le \operatorname{ess\,inf}_{\mathbb{R}^n\times(0,T)} u$. Then there exists a constant $\tau \in (0,1)$, depending only on a, L, μ_-, λ_-, θ and the data, such that if*

$$|\{u \le \mu_- + L\} \cap \mathcal{Q}^-(x_0,t_0;r,\theta)| \le \tau |\mathcal{Q}^-(x_0,t_0;r,\theta)|,$$

then

$$u \ge \mu_- + aL \ \textit{ a.e. in } \mathcal{Q}^-\big(x_0,t_0;\tfrac{3r}{4},\theta\big).$$

Proof. First we assume that $sp < n$. Without loss of generality, we may assume that $(x_0, t_0) = (0, 0)$. For $j = 0, 1, 2, \dots$, let

$$k_j = \mu_- + aL + \frac{(1-a)L}{2^j}, \quad \bar{k}_j = \frac{k_j + k_{j+1}}{2}, \quad r_j = \frac{3r}{4} + \frac{r}{2^{j+2}}, \quad \bar{r}_j = \frac{r_j + r_{j+1}}{2}, \tag{4.1}$$

$$B_j = B_{r_j}(0), \quad \overline{B}_j = B_{\bar{r}_j}(0), \quad w_j = (k_j - u)_+, \quad \bar{w}_j = (\bar{k}_j - u)_+, \tag{4.2}$$

and

$$\mathcal{Q}_j = B_j \times (-\theta r_j^{ps}, 0], \quad \overline{\mathcal{Q}}_j = \overline{B}_j \times (-\theta \bar{r}_j^{ps}, 0]. \tag{4.3}$$

We observe that $\mathcal{Q}_{j+1} \subset \overline{\mathcal{Q}}_j \subset \mathcal{Q}_j$, $\bar{k}_j < k_j$ and hence $\bar{w}_j \le w_j$. Moreover, from Lemma 4.1, we have

$$\frac{1}{\lambda(q)}(|u| + |k_j|)^{q-1} w_j^2 \le \xi(w_j) \le \lambda(q)(|u| + |k_j|)^{q-1} w_j^2. \tag{4.4}$$

Let $\psi_{1,j} \in C^\infty(B_j)$ and $\eta_{1,j} \in C^\infty((-\theta r_j^{ps}, 0])$ be nonnegative functions satisfying

$$0 \le \psi_{1,j} \le 1, \quad \psi_{1,j} = 1 \text{ in } B_{j+1}, \quad \operatorname{dist}(\operatorname{supp}\psi_{1,j}, \mathbb{R}^n \setminus B_j) \ge 2^{-j-1} r$$

and $0 \le \eta_{1,j} \le 1$, $\eta_{1,j} = 1$ on $(-\theta \bar{r}_j^{ps}, 0]$. Then for $\psi_j = \psi_{1,j}\, \eta_{1,j}$ we have

$$0 \le \psi_j \le 1 \text{ in } \mathcal{Q}_j, \quad \psi_j = 1 \text{ in } \overline{\mathcal{Q}}_j, \quad \psi_j = 0 \text{ on } \partial_p \mathcal{Q}_j,$$

and

$$|\nabla \psi_j| \le C \frac{2^j}{r}, \quad |\partial_t \psi_j| \le C \frac{2^{jps}}{\theta r^{ps}},$$

for some constant $C = C(n, p, s)$.

By Lemma 3.2 applied to w_j, there exists a constant $C = C(p, \Lambda)$, such that

$$\begin{aligned}
&\int_{-\theta {r_j}^{ps}}^{0} \int_{B_j} \int_{B_j} |w_j(x,t)\psi_j(x,t) - w_j(y,t)\psi_j(y,t)|^p \, d\mu \, dt \\
&\qquad + \operatorname*{ess\,sup}_{-\theta {r_j}^{ps} < t < 0} \int_{B_j} \psi_j(x,t)^p \xi(w_j(x,t)) \, dx \\
&\le C \Bigg(\int_{-\theta {r_j}^{ps}}^{0} \int_{B_j} \int_{B_j} \max\{w_j(x,t), w_j(y,t)\}^p |\psi_j(x,t) - \psi_j(y,t)|^p \, d\mu \, dt \\
&\qquad + \operatorname*{ess\,sup}_{(x,t) \in \operatorname{supp}\psi_j,\, -\theta r_j^{ps} < t < 0} \int_{\mathbb{R}^n \setminus B_j} \frac{w_j(y,t)^{p-1}}{|x-y|^{n+ps}} \, dy \int_{-\theta r_j^{ps}}^{0} \int_{B_j} w_j(x,t)\psi_j(x,t)^p \, dx \, dt \\
&\qquad + \int_{-\theta r_j^{ps}}^{0} \int_{B_j} \xi(w_j(x,t)) \partial_t \psi_j(x,t)^p \, dx \, dt \Bigg).
\end{aligned} \tag{4.5}$$

Now using (4.4), the fact that $\bar{w}_j \le w_j$ along with the properties of ψ_j in (4.5), we obtain

$$\begin{aligned}
&\int_{-\theta \bar{r}_j^{ps}}^{0} \int_{\overline{B}_j} \int_{\overline{B}_j} |\bar{w}_j(x,t) - \bar{w}_j(y,t)|^p \, d\mu \, dt + \operatorname*{ess\,sup}_{-\theta \bar{r}_j^{ps} < t < 0} \int_{\overline{B}_j} (|u| + |k_j|)^{q-1} \bar{w}_j^2 \, dx \\
&\le C(n, p, q, s, \Lambda)(I_1 + I_2 + I_3),
\end{aligned} \tag{4.6}$$

where

$$I_1 = \int_{-\theta {r_j}^{ps}}^{0} \int_{B_j} \int_{B_j} \max\{w_j(x,t), w_j(y,t)\}^p |\psi_j(x,t) - \psi_j(y,t)|^p \, d\mu \, dt,$$

$$I_2 = \operatorname*{ess\,sup}_{(x,t)\in \operatorname{supp}\psi_j,\, -\theta r_j^{ps}<t<0} \int_{\mathbb{R}^n\setminus B_j} \frac{w_j(y,t)^{p-1}}{|x-y|^{n+ps}}\, dy \int_{-\theta r_j^{ps}}^{0} \int_{B_j} w_j(x,t)\psi_j(x,t)^p \, dx\, dt, \text{ and}$$

$$I_3 = \int_{-\theta r_j^{ps}}^{0} \int_{B_j} (|u| + |k_j|)^{q-1} w_j^2 |\partial_t \psi_j(x,t)^p| \, dx\, dt.$$

Let $A_j = \{u < k_j\} \cap \mathcal{Q}_j$. Since $\mu_- \leq \operatorname{ess\,inf}_{\mathcal{Q}_r(x_0,t_0;r,\theta)} u$ and $\lambda_- \leq \operatorname{ess\,inf}_{\mathbb{R}\times(0,T)} u$, we have

$$w_j \leq \bar{L} := (\mu_- + L - \lambda_-)_+ \text{ in } \mathbb{R}^n \times (0,T). \tag{4.7}$$

We estimate the terms I_1 , I_2 and I_3 separately.

Estimate of I_1: Using the facts that $\frac{r}{2} < r_j < r$, (4.7) and the derivative bounds for ψ_j, we obtain

$$\begin{aligned} I_1 &= \int_{-\theta {r_j}^{ps}}^{0} \int_{B_j} \int_{B_j} \max\{w_j(x,t), w_j(y,t)\}^p |\psi_j(x,t) - \psi_j(y,t)|^p \, d\mu \, dt \\ &\leq C \frac{2^{jp}}{r^{ps}} \bar{L}^p |A_j|, \end{aligned} \tag{4.8}$$

where $C = C(n,p,s,\Lambda)$.

Estimate of I_2: For every $x \in \operatorname{supp}\psi_{1,j}$ and every $y \in \mathbb{R}^n \setminus B_j$, we have

$$\frac{1}{|x-y|} = \frac{1}{|y|}\frac{|x-(x-y)|}{|x-y|} \leq \frac{1}{|y|}(1+2^{j+3}) \leq \frac{2^{j+4}}{|y|}. \tag{4.9}$$

Using $r_j > \frac{r}{2}$, $0 \leq \psi_j \leq 1$ along with (4.7), we obtain

$$\begin{aligned} I_2 &\leq \operatorname*{ess\,sup}_{x\in \operatorname{supp}\psi_{1,j},\, -\theta r_j^{ps}<t<0} \int_{\mathbb{R}^n\setminus B_j} \frac{w_j(y,t)^{p-1}}{|x-y|^{n+ps}}\, dy \int_{-\theta r_j^{ps}}^{0} \int_{B_j} w_j(x,t)\, dx\, dt \\ &\leq C \frac{2^{j(n+ps)}}{r^{ps}} \bar{L}^p |A_j|, \end{aligned} \tag{4.10}$$

where $C = C(n,p,s)$.

Estimate of I_3: When $0 < q \leq 1$, using the inequality $|u| + |k_j| \geq w_j$ along with (4.7), we get

$$\begin{aligned} I_3 &= \int_{-\theta r_j^{ps}}^{0} \int_{B_j} (|u| + |k_j|)^{q-1} w_j^2 |\partial_t \psi_j(x,t)^p| \, dx\, dt \\ &\leq C \frac{2^{jps}}{\theta r^{ps}} \int_{-\theta r_j^{ps}}^{0} \int_{B_j} w_j^{q+1} \, dx\, dt \\ &\leq C \frac{2^{jps}}{\theta r^{ps}} \bar{L}^{q+1} |A_j|, \end{aligned} \tag{4.11}$$

where $C = C(n,p,q,s)$. For $q \geq 1$, we have $\mu_- \leq u \leq k_j \leq \mu_- + L$ in A_j. Hence, we get

$$I_3 = \int_{-\theta r_j^{ps}}^{0} \int_{B_j} (|u| + |k_j|)^{q-1} w_j^2 |\partial_t \psi_j(x,t)^p| \, dx \, dt \leq C \frac{2^{jps}}{\theta r^{ps}} M^{q-1} \bar{L}^2 |A_j|, \tag{4.12}$$

where $C = C(n,p,q,s)$ and $M = \max\{|\mu_-|, |\mu_- + L|\}$. Therefore, we have

$$I_3 \leq C \frac{2^{jps}}{r^{ps}} \frac{\max\{M^{q-1}, \bar{L}^{q-1}\} \bar{L}^2}{\theta} |A_j|, \tag{4.13}$$

where $C = C(n,p,q,s)$. Since $u \leq \bar{k}_j$, we have

$$(1-a)2^{-j-2} L = k_j - \bar{k}_j \leq k_j - u \leq |u| + |k_j| \leq 2M.$$

Therefore, for any $q > 0$, we get

$$(|u| + |k_j|)^{q-1} \bar{w}_j^2 \geq \frac{\gamma \min\{M^{q-1}, L^{q-1}\}}{2^{q(j+3)}} \bar{w}_j^2, \tag{4.14}$$

where

$$\gamma = \gamma(a,q) = \begin{cases} 2^{q-1}, & 0 < q < 1, \\ (1-a)^{q-1}, & q \geq 1. \end{cases} \tag{4.15}$$

By inserting (4.8), (4.10), (4.13) and (4.14) into (4.6), we obtain

$$\begin{aligned} &\int_{-\theta \bar{r}_j^{ps}}^{0} \int_{\overline{B}_j} \int_{\overline{B}_j} \frac{|\bar{w}_j(x,t) - \bar{w}_j(y,t)|^p}{|x-y|^{n+ps}} \, dx \, dy \, dt \\ &\qquad + \frac{\gamma \min\{M^{q-1}, L^{q-1}\}}{2^{q(j+3)}} \operatorname*{ess\,sup}_{-\theta \bar{r}_j^{ps} < t < 0} \int_{\overline{B}_j} \bar{w}_j^2 \, dx \\ &\quad \leq C \frac{2^{j(n+ps+p)}}{r^{ps}} \bar{L}^p \left(1 + \frac{\max\{M^{q-1}, \bar{L}^{q-1}\} \bar{L}^{2-p}}{\theta}\right) |A_j|, \end{aligned} \tag{4.16}$$

where $C = C(n,p,q,s,\Lambda)$. By using the inequality in Lemma 2.4 with $l = p(1 + \frac{2s}{n})$, we get

$$\int_{\overline{\mathcal{Q}}_j} |\bar{w}_j|^{p(1+\frac{2s}{n})} \, dx \, dt \leq \int_{-\theta \bar{r}_j^{ps}}^{0} \left(\int_{\overline{B}_j} |\bar{w}_j|^{p\kappa^*} \, dx \right)^{\frac{1}{\kappa^*}} dt \left(\operatorname*{ess\,sup}_{-\theta \bar{r}_j^{ps} < t < 0} \int_{\overline{B}_j} |\bar{w}_j|^2 \, dx \right)^{\frac{ps}{n}}. \tag{4.17}$$

By the Sobolev inequality in Lemma 2.3, there exists a constant $C = C(n,p,s)$ such that

$$\left(\fint_{\overline{B}_j} |\bar{w}_j|^{p\kappa^*} \, dx \right)^{\frac{1}{\kappa^*}} \leq C \bar{r}_j^{\frac{n}{\kappa^*}} \left(\bar{r}_j^{ps-n} \int_{\overline{B}_j} \int_{\overline{B}_j} \frac{|\bar{w}_j(x,t) - \bar{w}_j(y,t)|^p}{|x-y|^{n+ps}} \, dx dy + \fint_{\overline{B}_j} |\bar{w}_j|^p \, dx \right). \tag{4.18}$$

Noting that $\mathcal{Q}_{j+1} \subset \overline{\mathcal{Q}}_j \subset \mathcal{Q}_j$ and using (4.17) and (4.18), we obtain

$$\begin{aligned}
&\frac{(1-a)L}{2^{j+2}}|A_{j+1}| = \int_{A_{j+1}} (\bar{k}_j - k_{j+1})\,dxdt \le \int_{\mathcal{Q}_{j+1}} \bar{w}_j\,dxdt \le \int_{\overline{\mathcal{Q}}_j} \bar{w}_j\,dxdt \\
&\le \Big(\int_{\overline{\mathcal{Q}}_j} |\bar{w}_j|^{p(1+\frac{2s}{n})}\,dxdt\Big)^{\frac{n}{p(n+2s)}} |A_j|^{1-\frac{n}{p(n+2s)}} \\
&\le C\Bigg(\bar{r}_j^{\frac{n}{\kappa^*}} \bar{r}_j^{ps-n} \int_{-\theta\bar{r}_j^{ps}}^0 \int_{\overline{B}_j}\int_{\overline{B}_j} \frac{|\bar{w}_j(x,t)-\bar{w}_j(y,t)|^p}{|x-y|^{n+ps}}\,dx\,dy + \bar{r}_j^{\frac{n}{\kappa^*}} \int_{-\theta\bar{r}_j^{ps}}^0 ⨍_{\overline{B}_j} |\bar{w}_j|^p\,dxdt\Bigg)^{\frac{n}{p(n+2s)}} \\
&\qquad \cdot \Big(\operatorname*{ess\,sup}_{-\theta\bar{r}_j^{ps}<t<0} \int_{\overline{B}_j} |\bar{w}_j|^2\,dx\Big)^{\frac{s}{n+2s}} |A_j|^{1-\frac{n}{p(n+2s)}},
\end{aligned} \tag{4.19}$$

for some constant $C = C(n,p,s)$. By inserting (4.16) into (4.19), we get

$$\begin{aligned}
&\frac{(1-a)L}{2^{j+2}}|A_{j+1}| \\
&\le C\Bigg(\bar{r}_j^{\frac{n}{\kappa^*}}\Bigg(\bar{r}_j^{ps-n}\frac{2^{j(n+ps+p)}}{r^{ps}}\bar{L}^p\Big(1+\frac{\max\{M^{q-1},\bar{L}^{q-1}\}\bar{L}^{2-p}}{\theta}\Big)|A_j| + \bar{r}_j^{-n}L^p|A_j|\Bigg)\Bigg)^{\frac{n}{p(n+2s)}} \\
&\cdot\Bigg(\frac{2^{j(n+ps+p+q)}}{r^{ps}\gamma\min\{M^{q-1},L^{q-1}\}}\bar{L}^p\Big(1+\frac{\max\{M^{q-1},\bar{L}^{q-1}\}\bar{L}^{2-p}}{\theta}\Big)|A_j|\Bigg)^{\frac{s}{n+2s}} |A_j|^{1-\frac{n}{p(n+2s)}},
\end{aligned} \tag{4.20}$$

for some constant $C = C(n,p,q,s,\Lambda)$. Now since $\frac{r}{2} < \bar{r}_j < r$, we obtain from (4.20) that

$$\begin{aligned}
|A_{j+1}| \le{}& \frac{C}{(1-a)L\,r^{\frac{s(n+ps)}{n+2s}}}\Bigg(\bar{L}^p\Big(1+\frac{\max\{M^{q-1},\bar{L}^{q-1}\}\bar{L}^{2-p}}{\theta}\Big)\Bigg)^{\frac{n}{p(n+2s)}} \\
&\cdot\Bigg(\frac{\bar{L}^p}{\gamma\min\{M^{q-1},L^{q-1}\}}\Big(1+\frac{\max\{M^{q-1},\bar{L}^{q-1}\}\bar{L}^{2-p}}{\theta}\Big)\Bigg)^{\frac{s}{n+2s}} \\
&\cdot 2^{j\left(\frac{n(n+ps+p)}{p(n+2s)}+\frac{s(n+ps+p+q)}{n+2s}+1\right)} |A_j|^{1+\frac{s}{n+2s}},
\end{aligned} \tag{4.21}$$

where $C = C(n,p,q,s,\Lambda)$. By dividing both sides of (4.21) by $|\mathcal{Q}_{j+1}|$ and noting that $|\mathcal{Q}_j| < 2^{ps+n}|\mathcal{Q}_{j+1}|$, and also that $r_j < r$, we obtain

$$\begin{aligned}
Y_{j+1} \le{}& \frac{C(n,p,q,s,\Lambda)}{(1-a)L}\Bigg(\bar{L}^p\Big(1+\frac{\max\{M^{q-1},\bar{L}^{q-1}\}\bar{L}^{2-p}}{\theta}\Big)\Bigg)^{\frac{n}{p(n+2s)}} \\
&\cdot\Bigg(\frac{\theta\bar{L}^p}{\gamma\min\{M^{q-1},L^{q-1}\}}\Big(1+\frac{\max\{M^{q-1},\bar{L}^{q-1}\}\bar{L}^{2-p}}{\theta}\Big)\Bigg)^{\frac{s}{n+2s}} \\
&\cdot 2^{j\left(\frac{n(n+ps+p)}{p(n+2s)}+\frac{s(n+ps+p+q)}{n+2s}+1\right)} |Y_j|^{1+\frac{s}{n+2s}},
\end{aligned} \tag{4.22}$$

where we denote $Y_j = \frac{|A_j|}{|\mathcal{Q}_j|}$. By choosing

$$c_0 = \frac{C(n,p,q,s,\Lambda)}{(1-a)L}\left(\bar{L}^p\Big(1+\frac{\max\{M^{q-1},\bar{L}^{q-1}\}\bar{L}^{2-p}}{\theta}\Big)\right)^{\frac{n}{p(n+2s)}}$$
$$\cdot\left(\frac{\theta\bar{L}^p}{\gamma\min\{M^{q-1},L^{q-1}\}}\Big(1+\frac{\max\{M^{q-1},\bar{L}^{q-1}\}\bar{L}^{2-p}}{\theta}\Big)\right)^{\frac{s}{n+2s}},$$
$$b = 2^{\frac{n(n+ps+p)}{p(n+2s)}+\frac{s(n+ps+p+q)}{n+2s}+1},\ \beta=\frac{s}{n+2s},\ \tau=c_0^{-\frac{1}{\beta}}b^{-\frac{1}{\beta^2}}\in(0,1),$$

Lemma 4.2 gives $Y_j\to 0$ as $j\to\infty$ if $Y_0\le\tau$. This implies that

$$u\ge\mu_-+aL \text{ a.e. in } \mathcal{Q}^-_{\frac{3r}{4}}(\theta).$$

In the case when $ps\ge n$, we first choose N large enough such that $N>ps$ and apply Lemma 2.3 with $\kappa^*=\frac{N}{N-ps}$ and Lemma 2.4 with $l=p(1+\frac{2s}{N})$ in the proof above. Then by repeating the rest of the arguments above, we obtain a similar nonlinear iterative inequality as in (4.22) with $\beta=\frac{s}{N+2s}$. The conclusion of the lemma thus follows in this case as well.

Proof. [Proof of Theorem 2.12] Let $\theta=1$. Now it is quite straightforward to see from the proof of Lemma 4.3 above that $\frac{3r}{4}$ can be replaced by γr for any $\gamma<1$ in which case τ changes to a possibly smaller constant depending on γ. This implies that there exists $\tau\in(0,1)$ depending on a,L,μ^- and the data such that

$$|[u\le\mu^-+L]\cap\mathcal{Q}_r(x_0,t_0)|\le\tau|\mathcal{Q}_r(x_0,t_0)|,$$

implies $u\ge\mu^-+aL$ almost everywhere in $\mathcal{Q}_{cr}(x_0,t_0)$, for some $c=c(n,p,s)\in(0,1)$. In particular u satisfies the property $(\mathcal{D})$ with respect to centered cylinders $\mathcal{Q}_r(x_0,t_0)$ and thus the desired conclusion follows by an application of Theorem 2.11.

We state our second De Giorgi type lemma using which one can repeat the arguments in [18] to prove Theorem 2.14.

Lemma 4.4 *Let $1<p<\infty$, q, $L>0$, $a\in(0,1)$ and u be a weak supersolution of (1.1) in $\Omega\times(0,T)$. Let $\mathcal{Q}^-(x_0,t_0;r,\theta)\Subset\Omega\times(0,T)$ and assume that $\mu_-\le\operatorname{ess\,inf}_{\mathcal{Q}_r(x_0,t_0;\theta)}u$, $\lambda_-\le\operatorname{ess\,inf}_{\mathbb{R}^n\times(0,T)}u$. Then there exists a constant $\theta>0$ depending only on a,L,μ_-,λ_- and the data such that, if t_0 is a Lebesgue instant and*

$$u(\cdot,t_0)\ge\mu_-+L \text{ a.e. in } B_r(x_0),$$

then

$$u\ge\mu_-+aL \text{ a.e. in } \mathcal{Q}^+\big(x_0,t_0;\frac{3r}{4},\theta\big).$$

Proof. We may again assume that $ps<n$, because for the case $ps\ge n$ we can then modify the arguments exactly the same way as in Lemma 4.3. Let $\theta>0$ be such that $\mathcal{Q}^+(x_0,t_0;r,\theta)=B_r(x_0)\times[t_0,t_0+\theta r^{ps})\Subset\Omega\times(0,T)$. The parameter θ will be chosen below.

Without loss of generality, we may assume that $(x_0,t_0)=(0,0)$. Let k_j, $\bar{k}_j$, r_j, $\bar{r}_j$, $B_j\,\overline{B}_j$, w_j, $\bar{w}_j$, $j=0,1,2,\dots$, be as in (4.1) and (4.2). In contrast with (4.3), here we consider the forward in time cylinders

$$\mathcal{Q}_j=B_j\times[0,\theta r_j^{ps})\quad\text{and}\quad\overline{\mathcal{Q}}_j=\overline{B}_j\times[0,\theta\bar{r}_j^{ps}).$$

Let $\psi_j(x,t)=\psi_j(x)$ be a smooth time independent function vanishing on ∂B_r such that $0\le\psi_j\le1$ in $\mathcal{Q}_j$, $\psi_j=1$ in $\overline{\mathcal{Q}}_j$,

$$|\nabla\psi_j|\le C\frac{2^j}{r}\quad\text{and}\quad\operatorname{dist}(\operatorname{supp}\psi_j,\,\mathbb{R}^n\setminus B_j)\ge2^{-j-1}r,$$

for some constant $C=C(n,p,s)$. By Lemma 3.2, we obtain a constant $C=C(p,\Lambda)$ such that

$$\begin{aligned}
&\int_0^{\theta r_j^{ps}}\int_{B_j}\int_{B_j}|w_j(x,t)\psi_j(x)-w_j(y,t)\psi_j(y)|^p\,d\mu\,dt+\operatorname*{ess\,sup}_{0<t<\theta {r_j}^{ps}}\int_{B_j}\psi_j(x)^p\xi(w_j(x,t))\,dx\\
&\le C\Bigg(\int_0^{\theta r_j^{ps}}\int_{B_j}\int_{B_j}\max\{w_j(x,t),w_j(y,t)\}^p|\psi_j(x)-\psi_j(y)|^p\,d\mu\,dt\\
&\qquad+\operatorname*{ess\,sup}_{x\in\operatorname{supp}\psi_j,\,0<t<\theta r_j^{ps}}\int_{\mathbb{R}^n\setminus B_j}\frac{w_j(y,t)^{p-1}}{|x-y|^{n+ps}}\,dy\int_0^{\theta r_j^{ps}}\int_{B_j}w_j(x,t)\psi_j(x)^p\,dx\,dt\\
&\qquad+\int_0^{\theta r_j^{ps}}\int_{B_j}\xi(w_j(x,t))\partial_t\psi_j(x)^p\,dx\,dt+\int_{B_j\times\{0\}}{\psi_j}^p\xi(w_j(x,t))\,dx\Bigg).
\end{aligned}\tag{4.23}$$

Since ψ_j is independent of t, the third term in the right hand side of (4.23) vanishes. Since $u(\cdot,0)\ge\mu_-+L$ almost everywhere in $B_r(0)$, we have that $u(\cdot,0)\ge\mu_-+L>k_j$ almost everywhere in B_j. Therefore the last term on the right-hand side of (4.23) vanishes. Proceeding similarly as in the proof of Lemma 4.3, we obtain from (4.23) that

$$\begin{aligned}
&\int_0^{\theta\bar{r}_j^{ps}}\int_{\overline{B}_j}\int_{\overline{B}_j}\frac{|\bar{w}_j(x,t)-\bar{w}_j(y,t)|^p}{|x-y|^{n+ps}}\,dx\,dy\,dt+\frac{\gamma\min\{M^{q-1},L^{q-1}\}}{2^{jq}}\operatorname*{ess\,sup}_{0<t<\theta\bar{r}_j^{ps}}\int_{\overline{B}_j}\bar{w}_j^2\,dx\\
&\quad\le C\frac{2^{j(n+ps+p)}}{r^{ps}}\bar{L}^p|A_j|,
\end{aligned}\tag{4.24}$$

where $C=C(n,p,q,s,\Lambda)$, $A_j=\{u<k_j\}\cap\mathcal{Q}_j$, γ as defined in (4.15), $M=\max\{|\mu_-|,|\mu_-+L|\}$ and $\bar{L}=(\mu_-+L-\lambda_-)_+$. Observing that $\mathcal{Q}_{j+1}\subset\overline{\mathcal{Q}}_j\subset\mathcal{Q}_j$ and, by arguing similarly as in Lemma 4.3, we conclude that

$$Y_{j+1}\le\frac{C\bar{L}^{\frac{n+ps}{n+2s}}}{(1-a)L}\Big(\frac{\theta}{\gamma\min\{M^{q-1},L^{q-1}\}}\Big)^{\frac{s}{n+2s}}2^{j\left(\frac{n(n+ps+p)}{p(n+2s)}+\frac{s(n+ps+p+q)}{n+2s}+1\right)}Y_j^{1+\frac{s}{n+2s}},\tag{4.25}$$

where $C=C(n,p,q,s,\Lambda)$ and $Y_j=\frac{|A_j|}{|\mathcal{Q}_j|}$. Let

$$d_0=\frac{C\bar{L}^{\frac{n+ps}{n+2s}}}{(1-a)L}\Big(\frac{1}{\gamma\min\{M^{q-1},L^{q-1}\}}\Big)^{\frac{s}{n+2s}},$$

where $C = C(n,p,q,s,\Lambda)$. By choosing

$$b = 2^{\left(\frac{n(n+ps+p)}{p(n+2s)}+\frac{s(n+ps+p+q)}{n+2s}+1\right)}, \quad \beta = \frac{s}{n+2s}, \quad c_0 = d_0\,\theta^{\beta}$$

in Lemma 4.2, we have $Y_j \to 0$ as $j \to \infty$, if $Y_0 \le \nu = c_0^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}}$. By choosing $\theta = \delta\, d_0^{-\frac{1}{\beta}}\, b^{-\frac{1}{\beta^2}}$ for $\delta \in (0,1)$, we get $\nu = \delta^{-1} > 1$. Hence the fact that $Y_0 \le 1$ and Lemma 4.2 imply that $Y_j \to 0$ as $j \to \infty$. Therefore, we have

$$u \ge \mu_- + aL \text{ a.e. in } \mathcal{Q}^+_{\frac{3r}{4}}(\theta).$$

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References

[1] Agnid Banerjee, Prashanta Garain, and Juha Kinnunen. Some local properties of subsolution and supersolutions for a doubly nonlinear nonlocal pLaplace equation. *Annali di Matematica Pura ed Applicata (1923 -), DOI: https://doi.org/10.1007/s10231-021-01177-4)*

[2] Verena Bögelein, Frank Duzaar, and Naian Liao. On the Hölder regularity of signed solutions to a doubly nonlinear equation. *J. Funct. Anal.*, 281: 109173, 58, 2021.

[3] Matteo Bonforte, Yannick Sire, and Juan Luis Vázquez. Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains. *Discrete Contin. Dyn. Syst.*, 35(12):5725–5767, 2015.

[4] Matteo Bonforte and Juan Luis Vázquez. Quantitative local and global a priori estimates for fractional nonlinear diffusion equations. *Adv. Math.*, 250:242–284, 2014.

[5] Lorenzo Brasco, Erik Lindgren and Armin Schikorra. Higher Hölder regularity for the fractional p-Laplacian in the superquadratic case. *Adv. Math.* 338 (2018), 782-846.

[6] Lorenzo Brasco, Erik Lindgren, and Martin Strömqvist. Continuity of solutions to a nonlinear fractional diffusion equation. *arXiv e-prints*, page arXiv:1907.00910, Jul 2019.

[7] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci. Local behavior of fractional p-minimizers. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(5):1279–1299, 2016.

[8] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci. Nonlocal Harnack inequalities. *J. Funct. Anal.*, 267(6):1807–1836, 2014.

[9] Emmanuele DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.

[10] Matthieu Felsinger and Moritz Kassmann. Local regularity for parabolic nonlocal operators. *Comm. Partial Differential Equations*, 38(9):1539–1573, 2013.

[11] Ugo Gianazza and Vincenzo Vespri. A Harnack inequality for solutions of doubly nonlinear parabolic equations. *J. Appl. Funct. Anal.*, 1(3):271–284, 2006.

[12] Moritz Kassmann. A new formulation of Harnack's inequality for nonlocal operators. *C. R. Math. Acad. Sci. Paris*, 349(11-12):637–640, 2011.

[13] Juha Kinnunen and Peter Lindqvist. Definition and properties of supersolutions to the porous medium equation. *J. Reine Angew. Math.*, 618:135–168, 2008.

[14] Juha Kinnunen and Peter Lindqvist. Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Mat. Pura Appl. (4)*, 185(3):411–435, 2006.

[15] Janne Korvenpää, Tuomo Kuusi, and Giampiero Palatucci. Fractional superharmonic functions and the Perron method for nonlinear integro-differential equations. *Math. Ann.*, 369(3-4):1443–1489, 2017.

[16] Tuomo Kuusi. Lower semicontinuity of weak supersolutions to nonlinear parabolic equations. *Differential Integral Equations*, 22(11-12):1211–1222, 2009.

[17] Tuomo Kuusi, Juhana Siljander, and José Miguel Urbano. Local Hölder continuity for doubly nonlinear parabolic equations. *Indiana Univ. Math. J.*, 61(1):399–430, 2012.

[18] Naian Liao. Regularity of weak supersolutions to elliptic and parabolic equations: lower semicontinuity and pointwise behavior. *J. Math. Pures Appl. (9)*, 147: 179–204, 2021.

[19] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[20] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez. A fractional porous medium equation. *Adv. Math.*, 226(2):1378–1409, 2011.

[21] Martin Strömqvist. Harnack's inequality for parabolic nonlocal equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(6):1709–1745, 2019.

[22] Martin Strömqvist. Local boundedness of solutions to non-local parabolic equations modeled on the fractional p-Laplacian. *J. Differential Equations*, 266(12):7948–7979, 2019.

[23] William P. Ziemer. Regularity of weak solutions of parabolic variational inequalities. *Trans. Amer. Math. Soc.*, 309(2):763–786, 1988.

Agnid Banerjee
Tata Institute of Fundamental Research
Centre for Applicable Mathematics
Bangalore-560065, India
Email: agnidban@gmail.com

Prashanta Garain
Department of Mathematics
Ben Gurion University of the Negev
Beer Sheva-84105, Israel
Email: pgarain92@gmail.com

Juha Kinnunen
Department of Mathematics
P.O. Box-11100
FI-00076 Aalto University, Finland
Email: juha.k.kinnunen@aalto.fi