Waves in Linear Time-Varying Dielectric Media

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Abstract—In this paper, focusing on the frequency domain, we write the constitutive relation and the Helmholtz equation for linear, dispersive, and inhomogeneous time-varying media. Next, by assuming spatial homogeneity, we simplify the equations and explain how to calculate dispersion curves (the angular frequency with respect to the wave vector) for propagating waves. Furthermore, we show that under the simplifying assumption of instantaneous response, the developed general approach provides the same dispersion curves as reported earlier for the dispersionless model of time-varying dielectric media. We believe that this study is important for investigations of wave phenomena in time-varying media, properly taking into account inevitable frequency dispersion of materials.

Index Terms—Dispersion curves, temporal complex permittivity, time-varying dispersive media.

I. INTRODUCTION

In nature, we see temporal nonlocality as an inherent characteristic of any material. Specifically, it means that reaction, such as the induced polarization density, always experiences delay with respect to action, such as the electric or magnetic field. Combining with causality, we assert that the reaction depends on the action only in the past time, and the delay time between the reaction and the action is always positive. In electromagnetic theory, for a linear and time-invariant dielectric medium, this fundamental principle defines the conventional convolution relation, in which the polarization density is found as the convolution of the response function (the susceptibility kernel) and the electric field:

$$P(r,t) = \epsilon_0 \int_{0}^{+\infty} \hat{\chi}(\gamma)E(r,t-\gamma) d\gamma.$$  

Indeed, the above convolution integral properly models causal and nonlocal response in time (see, e.g., Ref. [1]). For simplicity of studying wave phenomena, we use the Fourier transform and readily write the above expression in the frequency domain as

$$P(r,\omega) = \epsilon_0 \hat{\chi}(\omega)E(r,\omega).$$  

One intriguing curiosity is what happens if the medium is kept linear but becomes not temporally immutable. For example, the number of atoms per unit volume or such material parameters as the damping coefficient or resonance frequencies of atoms response vary in time. In this case, the invariance under time translation is certainly broken, and the medium becomes nonstationary (in other words, if the action shifts in time, the reaction does not shift by the same amount of time). Foundational questions arise: How does the constitutive relation change in the frequency domain? And how does this change affect the dispersion curves for plane-wave solutions of Maxwell’s equations?

In this presentation, we are going to answer these questions. For that, we apply the definitions first introduced by Zadeh in his 1950 paper [2] describing time-variable networks. In the context of electrodynamics, Stepanov used the same definitions for deriving the susceptibility of a plasma whose electron density changes in time [3]. However, in the work by Stepanov, the polarization density as a response to the time-harmonic electric field is not finally described purely in the frequency domain. Secondly, he only wanted to calculate the proper susceptibility corresponding to time-varying plasma, and there was no discussion about the Helmholtz equation and dispersion curves. Here, by considering any arbitrary temporal function for the electric field, we use the constitutive relation which is expressed purely in the frequency domain. Importantly, this approach allows us to derive the Helmholtz equation that finally provides us the dispersion curves for plane-wave solutions. We stress that this work, as a continuation of our previous efforts [4], [5], [6] and other works such as [7] to understand time-varying (artificial) materials and meta-atoms, is presenting a general tool that correctly describes any linear
and nonstationary medium which is homogenized and modeled by an effective macroscopic susceptibility or permittivity (see, Fig. 1).

II. GENERAL APPROACH

Interactions of electromagnetic waves with a medium are described based on the eigenmode solutions. These solutions can be expressed in terms of the dispersion relation and the corresponding dispersion curves which show the angular frequency with respect to the wave vector of propagating waves. For deriving the dispersion relation, first, we need the constitutive relations that connect the electric and magnetic density fluxes to the electric and magnetic fields. Next, by applying Maxwell’s equations, we derive the Helmholtz equation in the frequency domain. Solving this equation gives a link between the angular frequency and the wave vectors corresponding to propagating waves. Thus, let us start by writing the constitutive relation in the frequency domain.

A. Constitutive relation

For a space-time-varying dielectric medium which is linear and causal, we write the general nonlocal relation between the electric flux density \( \mathbf{D} \) and the electric field \( \mathbf{E} \) as

\[
\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \varepsilon_0 \int_{0}^{\infty} \left[ \int_{V} \hat{\chi}(\mathbf{r}', \mathbf{r}, \gamma, t) \mathbf{E}(\mathbf{r}' - \mathbf{r}, t - \gamma) d\mathbf{r}' \right] d\gamma.
\]  

(3)

where \( \hat{\chi} \) denotes the susceptibility kernel which depends on two position vectors \( (\mathbf{r}, \mathbf{r}') \) and two time variables \( (t, \gamma) \). Regarding the spatial characteristic, the above equation means that 1) the medium is spatially inhomogeneous and 2) the electric flux density at each point in space depends on the electric field at other locations. However, for avoiding difficulty, let us assume that the response is local in space, i.e., neglect spatial dispersion, and assume that \( \hat{\chi}(\mathbf{r}', \mathbf{r}, \gamma, t) = \delta^3(\mathbf{r}') \hat{\chi}(\mathbf{r}, \gamma, t) \), in which \( \delta^3(\mathbf{r}) \) is the three-dimensional Dirac delta function. Under this assumption, Eq. (3) is simplified as

\[
\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \varepsilon_0 \int_{0}^{\infty} \hat{\chi}(\mathbf{r}, \gamma, t) \mathbf{E}(\mathbf{r}, t - \gamma) d\gamma.
\]  

(4)

Here, the first time variable, \( \gamma \), is the delay time between the polarization density and the electric field (modeling the frequency dispersion effects). The second variable, \( t \), is the observation time (representing the nonstationary property of the medium). In the above equation, electric field at each point can be an arbitrary function of time. Therefore, we can write the electric field as the inverse Fourier transform of \( \mathbf{E}(\mathbf{r}, \omega) \):

\[
\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) \exp(j\omega t) d\omega.
\]  

(5)

We replace the electric field in the constitutive relation, Eq. (4), by the above expression. Consequently, we achieve an important expression for the instantaneous electric flux density, which reads

\[
\mathbf{D}(\mathbf{r}, t) = \frac{\varepsilon_0}{2\pi} \int_{-\infty}^{\infty} \hat{\mathbf{e}}_T(\mathbf{r}, \omega, t) \mathbf{E}(\mathbf{r}, \omega) \exp(j\omega t) d\omega.
\]  

(6)

Here, \( \hat{\mathbf{e}}_T(\mathbf{r}, \omega, t) = 1 + \hat{\chi}_T(\mathbf{r}, \omega, t) \) and

\[
\hat{\chi}_T(\mathbf{r}, \omega, t) = \int_{0}^{\infty} \hat{\chi}(\mathbf{r}, \gamma, t) \exp(-j\omega \gamma) d\gamma
\]  

(7)

is the Fourier transform of the susceptibility kernel with respect to the delay time. We name this function as temporal complex susceptibility. According to Eq. (6), we explicitly observe that by taking the second Fourier transform, we fully exit from the time domain, and we can investigate wave-matter interactions utterly in the frequency domain. However, first, this second Fourier transform should be with respect to the time variable \( t \). Second, we must apply a different angular frequency notation rather than \( \omega \). Here, we use the capital letter \( \Omega \) in the Greek alphabet for this purpose. Knowing that \( \exp(j\omega t) \) gives rise to a shift in the frequency domain, we conclude that

\[
\mathbf{D}(\mathbf{r}, \Omega) = \frac{\varepsilon_0}{2\pi} \int_{-\infty}^{\infty} \hat{\mathbf{e}}_T(\mathbf{r}, \omega, \Omega - \omega) \mathbf{E}(\mathbf{r}, \omega) d\omega.
\]  

(8)

B. Helmholtz equation

Deriving the constitutive relation in the frequency domain, we take the second step and substitute this equation into Maxwell’s equations in order to obtain the Helmholtz equation. As usually, we consider eigenwaves in an unbounded source-free dielectric medium. Accordingly, we write that

\[
\nabla \times \mathbf{E}(\mathbf{r}, \Omega) = -j\Omega \mu_0 \mathbf{H}(\mathbf{r}, \Omega),
\]

\[
\nabla \times \mathbf{H}(\mathbf{r}, \Omega) = j\Omega \mu_0 \mathbf{E}(\mathbf{r}, \Omega),
\]

\[
\nabla \cdot \mathbf{D}(\mathbf{r}, \Omega) = 0,
\]

\[
\nabla \cdot \mathbf{B}(\mathbf{r}, \Omega) = 0.
\]

(9)

We take the curl from both sides of the Faraday law. Using a basic vector algebra relation \( \nabla \times \nabla \times \mathbf{V} = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} \), in which \( \mathbf{V} \) is an arbitrary differentiable vector field in space, we obtain \( \nabla \cdot \left( \nabla \times \mathbf{E}(\mathbf{r}, \Omega) \right) - \nabla^2 \mathbf{E}(\mathbf{r}, \Omega) = -j\Omega \mu_0 \nabla \times \mathbf{H}(\mathbf{r}, \Omega) \). On the other hand, curl of \( \mathbf{H} \) is related to the electric flux density through the Ampere-Maxwell law, where \( \mathbf{D} \) is related to \( \mathbf{E} \) by the constitutive relation. Therefore, we deduce that

\[
\nabla \left( \nabla \cdot \mathbf{E}(\mathbf{r}, \Omega) \right) - \nabla^2 \mathbf{E}(\mathbf{r}, \Omega) - \Omega^2 \mu_0 \varepsilon_0 \int_{-\infty}^{\infty} \hat{\mathbf{e}}_T(\mathbf{r}, \omega, \Omega - \omega) \mathbf{E}(\mathbf{r}, \omega) d\omega = 0.
\]  

(10)

However, beside this equation, there is another equation that is based on the Gauss law stating that the divergence of the electric flux density should be zero. Let us recall that \( \nabla \cdot \left( \nabla \times \mathbf{V} \right) = \nabla \cdot \mathbf{J} \). Here, \( \mathbf{J} \) is an arbitrary differentiable scalar function. By using this algebraic rule, we find the second equation in form

\[
\int_{-\infty}^{\infty} \mathbf{e}_T(\mathbf{r}, \omega, \Omega - \omega) \nabla \cdot \mathbf{E}(\mathbf{r}, \omega) d\omega = - \int_{-\infty}^{\infty} \nabla \mathbf{e}_T(\mathbf{r}, \omega, \Omega - \omega) \cdot \mathbf{E}(\mathbf{r}, \omega) d\omega.
\]  

(11)

These two equations, Eqs. (10) and (11), determine the possible wave vectors of propagating waves. In the above result,
what is intriguing is the presence of the gradient of permittivity, and, simultaneously, the dependency of permittivity on two angular frequencies. These features may result in interesting possibilities for wave manipulation.

From now, let us assume that the medium is effectively homogeneous in space. In this case, the gradient of permittivity vanishes, and we have

$$\int_{-\infty}^{+\infty} e_T(\omega, \Omega - \omega) \nabla \cdot E(r, \omega) d\omega = 0. \quad (12)$$

At this point, we suppose that the electric field is written as the multiplication of two functions: $E(r, \Omega) = R(r)G(\Omega)$ (as we will see later, this assumption is valid, and the general solution for the field is a summation of all possible eigenfunctions which are written as the multiplication of the two mentioned functions). If we substitute this expression in Eq. (12), we simply observe that the term $\nabla \cdot R(r)$ comes out from the integral, and two possibilities arise. The first possibility is that $\nabla \cdot E = 0$ meaning that $\nabla \cdot \mathbf{E} = 0$. Accordingly, Eq. (10) is simplified, and we finally infer that

$$\nabla^2 E(r, \Omega) + \frac{\Omega^2 \mu_0 \epsilon_0}{2\pi} \int_{-\infty}^{+\infty} e_T(\omega, \Omega - \omega) E(r, \omega) d\omega = 0. \quad (13)$$

This is the Helmholtz equation for dispersive time-varying media. As a quick check, for the conventional time-invariant case, the temporal complex permittivity is equal to $e_T(\omega, t) = \epsilon(\omega)$. Thus, the Fourier transform gives rise to $e_T(\omega, \Omega) = 2\pi \epsilon(\omega) \delta(\Omega)$. Substituting this relation into Eq. (13) and knowing that $\int_{-\infty}^{+\infty} f(x) \delta(x - x_0) dx = f(x_0)$, we obtain

$$\nabla^2 E(r, \Omega) + k^2 E(r, \Omega) = 0 \quad (k^2 = \Omega^2 \mu_0 \epsilon_0(\Omega)))$$

which is the classical text-book result [8].

C. Special scenario: Instantaneous response

In the following, let us first focus on the special case where the medium can be assumed to be non dispersive (which is a dramatic simplifying approximation) to prove that the general theory, developed above, gives the same results as those previously derived in other works under the same simplifying assumption. In the conference talk, we will show the results for the general case that takes temporal dispersion into account.

If the temporal dispersion is neglected, we express the permittivity kernel as $\epsilon(\gamma, t) = \epsilon(t) \delta(\gamma)$ which results in $\epsilon_T(\omega, t) = \epsilon(t)$. By using Eq. (6), we see that $D(t) = \epsilon(t) E(t)$. Now, let us limit ourselves to periodic modulations of permittivity. In this case, we can write $\epsilon(t)$ based on the Fourier series as

$$\epsilon(t) = \sum_{n=-\infty}^{+\infty} \epsilon_n \exp(jn\Omega_p t), \quad (14)$$

in which $\Omega_p$ represents the angular frequency related to the modulation period. Also, $\epsilon_n$ is associated with the Fourier coefficients. Using the above expression, we readily conclude that

$$\epsilon_T(\omega, \Omega) = \sum_{n=-\infty}^{+\infty} 2\pi \epsilon_n \delta(\Omega - n\Omega_p). \quad (15)$$

We need to substitute this expression as the Fourier transform of the temporal complex permittivity into Eq. (13). By doing this and keeping in mind the definition of the Dirac delta function (explained before), we achieve

$$\nabla^2 E(r, \Omega) + \Omega^2 \mu_0 \epsilon_0 \sum_{n=-\infty}^{+\infty} \epsilon_n E(r, \Omega - n\Omega_p) = 0. \quad (16)$$

Let us try to solve the above partial differential equation in Cartesian coordinates. We consider a plane wave traveling along the $z$-direction. Therefore, $\nabla^2$ and $r$ in Eq. (16) can be replaced by $\frac{\partial^2}{\partial z^2}$ and $z$, respectively. Also, we assume that the electric field can be written as the multiplication of two separated functions of $z$ and $\Omega$, i.e., $E(z, \Omega) = \sum_{l=1}^{+\infty} R_l(z)G_l(\Omega)$. The reason for having summation with respect to the integer index $l$ is the fact that it is possible that there are infinitely many solutions for eigenvalues. Therefore, the general solution for the electric field is the summation of all the possible solutions of this form. These possible solutions are linearly independent and form a basis to construct general solutions for electric field.

Substitution of the assumed form of electric field into Eq. (16) gives

$$\frac{\partial^2}{\partial z^2} \sum_{l=1}^{+\infty} R_l(z)G_l(\Omega)$$

$$\quad + \Omega^2 \mu_0 \epsilon_0 \sum_{n=-\infty}^{+\infty} \epsilon_n \sum_{l=1}^{+\infty} R_l(z)G_l(\Omega - n\Omega_p) = 0.$$  \quad (17)

Since $R_l(z)G_l(\Omega)$ are linearly independent, we can claim that Eq. (17) is valid if we ignore summation of $l$. Therefore, after doing a bit of simplification, we have the following equation for each possible solution for electric field:

$$\frac{1}{R_l(z)} \frac{\partial^2}{\partial z^2} R_l(z) + \frac{\Omega^2 \mu_0 \epsilon_0}{G_l(\Omega)} \sum_{n=-\infty}^{+\infty} \epsilon_n G_l(\Omega - n\Omega_p) = 0. \quad (18)$$

The first part of this equation is dependent on $z$ and the second one on $\Omega$. Thus, each term is equal to a constant value, $k_l^2$, but with opposite signs:

$$\frac{1}{R_l(z)} \frac{\partial^2}{\partial z^2} R_l(z) = -k_l^2$$  \quad (19)

and

$$\Omega^2 \mu_0 \epsilon_0 \sum_{n=-\infty}^{+\infty} \epsilon_n G_l(\Omega - n\Omega_p) = k_l^2 G_l(\Omega). \quad (20)$$

The solution of Eq. (19) can be expressed as

$$R_l(z) = f_l \exp(\pm jk_l z). \quad (21)$$

This equation tells that the electric field is a summation of harmonic waves with different wave numbers $k_l$. In order to
By substituting \( \Omega : \Omega \rightarrow n\Omega_p \), where \( \delta_{ij} \) is the Kronecker delta (if \( i = j \), it is equal to unity, otherwise it is zero). If we define \( n' : n + m \), we simply have
\[
\sum_{n' = -\infty}^{+\infty} \left[ \Omega^2 \mu_0 \varepsilon_0 \varepsilon_{n'-m} - k_l^2 \delta_{n'm} \right] G_l(\Omega - (n' - m)\Omega_p) = 0.
\]
(23)

By substituting \( \Omega : \Omega' = m\Omega_p \), one can rewrite the above equation as
\[
\sum_{n' = -\infty}^{+\infty} \left[ (\Omega' - m\Omega_p)^2 \mu_0 \varepsilon_0 \varepsilon_{n'-m} - k_l^2 \delta_{n'm} \right] G_l(\Omega' - n'\Omega_p) = 0.
\]
(24)

Here, we drop the primes to make the expression more clear. As we will see, Eq. (24) makes the procedure of solving the equation numerically simpler as compared to Eq. (22).

Therefore, finally, we infer that
\[
\sum_{n' = -\infty}^{+\infty} \left[ (\Omega' - m\Omega_p)^2 \mu_0 \varepsilon_0 \varepsilon_{n'-m} - k_l^2 \delta_{nm} \right] G_l(\Omega' - n'\Omega_p) = 0.
\]
(25)

This equation can be written in matrix form, where \( (\Omega - m\Omega_p)^2 \mu_0 \varepsilon_0 \varepsilon_{n'-m} \) is an infinite size square matrix which is multiplied by a vector formed by the values \( G_l(\Omega - n\Omega_p) \). Likewise, the expression \( k_l^2 \delta_{nm} \) can be considered as a matrix multiplied by the same vector. In the trivial case when there is no time modulation, so that \( \varepsilon_{n'-m} = \varepsilon_r \delta_{nm} \), the mentioned infinite size square matrix reduces to a diagonal matrix.

Therefore, here, we are dealing with an eigenvalue problem. \( k_l^2 \delta_{nm} \) are the eigenvalues and \( G_l(\Omega - n\Omega_p) \) are the eigenvectors. By solving this equation, we obtain the acceptable wave numbers for a specific angular frequency and the complete form of the electric field.

In order to study the properties of waves in nondispersive time-periodic media, we can solve this equation numerically. Here, we present results for two single and multiple harmonic modulations, see Fig. 2. It is evident that by modulating the permittivity of a nondispersive medium in time in a periodic manner, band-gaps in some ranges of \( k \) appear, and by adding more harmonics of modulation, we can open more gaps and control them.

The special case of a nondispersive time-varying medium with a single-harmonic modulation was studied in Ref. [9], leading to similar results for dispersion curves as we show here. We stress that if the medium is dispersive, the method in Ref. [9] cannot be used. In this general case of dispersive time-varying media, it is possible to use the general method outlined in this paper.

III. Conclusion

In this paper, we have developed a rigorous frequency-domain method to study wave propagation in inhomogeneous and homogeneous time-varying dielectric media. To do the study completely in the frequency domain, we have introduced two angular frequencies: \( \omega \) and \( \Omega \) (which is necessary because the time-domain susceptibility kernel for such a medium is a function of two time variables). The first frequency argument \( \omega \) is related to the temporal nonlocality (or temporal dispersion), and the second argument \( \Omega \) is associated with the observation time. By using the notion of temporal complex susceptibility (or permittivity) and its Fourier transform, we have shown how
the instantaneous and frequency-domain electric flux densities are connected to the electric field. Having the constitutive relation purely in the frequency domain, we applied Maxwell’s equations and finally achieved the Helmholtz equation which is solved to obtain the corresponding dispersion curves for eigensolutions in form of plane waves. For the special case of homogeneous nondispersive dielectric media with time-periodic modulations, we have confirmed that our results, obtained by using the general approach developed in this paper, are the same as the ones already reported in the literature for that simplified case. We are currently using the developed theory to understand scattering of waves from slabs made of dispersive time-varying dielectric media.

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REFERENCES